Multiplicity-Free Permutation Characters in GAP, part 2

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Abstract

We complete the classification of the multiplicity-free permutation actions of nearly simple groups that involve a sporadic simple group, which had been started in [BL96] and [LM].

Contents

1 Introduction

In [BL96], the multiplicity-free permutation characters of the sporadic simple groups and their automorphism groups were classified. Based on this list, the multiplicity-free permutation characters of the central extensions of the sporadic simple groups were classified in [LM].

The purpose of this writeup is to show how the multiplicity-free permutation characters of the automorphic extensions of the central extensions of the sporadic simple groups can be computed, to verify the calculations in [LM] (and to correct an error, see Section 3.32), and to provide a test file for the GAP functions and the database.

The database has been extended in the sense that also most of the character tables of the multiplicity-free permutation modules of the sporadic simple groups and their automorphic and central extensions have been computed, see [Höh01, Mül03, BM05, Mül08] for details.

Five errors in an earlier version (from July 2003) have been pointed out by Jürgen Müller. These errors concern the numbers of conjugacy classes of certain point stabilizers in $2.J_2.2$, $2.HS.2$, and $6.Fi_{22}.2$ (see Sections 3.12, 3.14, and 3.33).

The only differences between the current version and the version that was available since 2005 are additions of references, adjustments of group names in the data file, and adjustments of the GAP output format to version 4.5, see [GAP12].

Note that the version from 2003 was based on a data file that contained only the permutation character information, whereas the current version uses the database file of [BM05], which includes also the known character tables of endomorphism rings.

2 The Approach

Suppose that a group G contains a normal subgroup N. If π is a faithful multiplicity-free permutation character of G then $\pi = 1_U^G$ for a subgroup U of G that intersects N trivially, so π contains a constituent 1_{UN}^G of degree $\pi(1)/|N|$, which can be viewed as a multiplicity-free permutation character of the factor group G/N. Moreover, no constituent of the difference $\pi - 1_{UN}^G$ has N in its kernel.

So if we know all multiplicity-free permutation characters of the factor group G/N then we can compute all candidates for multiplicity-free permutation characters of G by "filling up" each such character $\overline{\pi}$ with a linear combination of characters not containing N in their kernels, of total degree $(|N| - 1) \cdot \pi(1)$, and such that the sum is a possible permutation character of G. For this situation, GAP provides a special variant of the function `PermChars`. In a second step, the candidates are inspected whether the required point stabilizers (and if yes, how many conjugacy classes of them) exist. Finally, the permutation characters are verified by explicit induction from the character tables of the point stabilizers.

The multiplicity-free permutation actions of the sporadic simple groups and their automorphism groups are known by [BL96], so this approach is suitable for these groups.

For central extensions of sporadic simple groups, the multiplicity-free permutation characters have been classified in [LM]; this note describes a slightly different approach, so we will give an independent confirmation of their results (except for the error pointed out in Section 3.32).

First we load the Character Table Library [Bre12] of the GAP system [GAP11], and the GAP interface (see [WPN+11]) to the ATLAS of Group Representations (see [WWT+]).

```
gap> LoadPackage( "ctbllib" );
true
gap> LoadPackage( "atlasrep" );
true
```

Then we read –if necessary– the file with GAP functions for computing multiplicity-free permutation characters, and the file with the data. Note that this includes the data we are going to compute, but we will actually **use** only the data for sporadic simple groups and their automorphism groups. For the other groups, we will compare the results computed below with the database.

```
gap> if not IsBound( PossiblePermutationCharactersWithBoundedMultiplicity )
>        then
>      ReadPackage( "ctbllib", "tst/multfree.g" );
>    fi;
gap> if not IsBound( MULTFREEINFO ) then
>      ReadPackage( "ctbllib", "tst/mferctbl.gap" );
>    fi;
gap> if not IsBound( PossiblePermutationCharactersWithBoundedMultiplicity ) or
>        not IsBound( MULTFREEINFO ) then
>      Print( "Sorry, the data files are not available!\n" );
>    fi;
```

(If the data files are not available then they can be fetched from the homepage of the GAP Character Table Library [Bre12].)

2.1 Computing Possible Permutation Characters

Next we define the GAP functions that are needed in the following.

The utility function `PossiblePermutationCharacters` takes two ordinary character tables `sub` and `tbl`, and returns the set of all induced class functions of the trivial character of `sub` to `tbl`, w.r.t. the possible class fusions from `sub` to `tbl`. (The entries in the result list are not necessarily multiplicity-free.)

```
gap> PossiblePermutationCharacters:= function( sub, tbl )
>    local fus, triv;
>
>    fus:= PossibleClassFusions( sub, tbl );
>    if fus = fail then
>      return fail;
>    fi;
>    triv:= [ TrivialCharacter( sub ) ];
>
>    return Set( List( fus, map -> Induced( sub, tbl, triv, map )[1] ) );
>    end;;
```

`FaithfulCandidates` takes the character table `tbl` of a group G and the name `factname` of a factor group F of G for which the multiplicity-free permutation characters are known, and returns a list of lists, the entry at the i-th position being the list of possible permutation characters of G that are multiplicity-free and such that the sum of all constituents that are characters of F is the i-th multiplicity-free permutation character of F. As a side-effect, if the i-th entry is nonempty then information is printed about the structure of the point-stabilizer in F and the number of candidates found.

```
gap> FaithfulCandidates:= function( tbl, factname )
>    local factinfo, factchars, facttbl, fus, sizeN, faith, i;
>
>    # Fetch the data for the factor group.
>    factinfo:= MultFreeEndoRingCharacterTables( factname );
>    factchars:= List( factinfo, x -> x.character );
>    facttbl:= UnderlyingCharacterTable( factchars[1] );
>    fus:= GetFusionMap( tbl, facttbl );
>    sizeN:= Size( tbl ) / Size( facttbl );
>
>    # Compute faithful possible permutation characters.
>    faith:= List( factchars, pi -> PermChars( tbl,
>                      rec( torso:= [ sizeN * pi[1] ],
>                           normalsubgroup:= ClassPositionsOfKernel( fus ),
>                           nonfaithful:= pi{ fus } ) ) );
>
>    # Take only the multiplicity-free ones.
>    faith:= List( faith, x -> Filtered( x, pi -> ForAll( Irr( tbl ),
>                      chi -> ScalarProduct( tbl, pi, chi ) < 2 ) ) );
>
>    # Print info about the candidates.
>    for i in [ 1 .. Length( faith ) ] do
>      if not IsEmpty( faith[i] ) then
```

```
>       Print( i, ":  subgroup ", factinfo[i].subgroup,
>              ", degree ", faith[i][1][1],
>              " (", Length( faith[i] ), " cand.)\n" );
>     fi;
>   od;
>
>   # Return the candidates.
>   return faith;
>   end;;
```

2.2 Verifying the Candidates

In the verification step, we check which of the given candidates of G are induced from a given subgroup S. For that, we use the following function. Its arguments are the character table `s` of S, the character tables `tbl2` and `tbl` of G and its derived subgroup G' of index 2 (if G is perfect then `0` must be entered for `tbl2`), the list `candidates` of characters of G, and one of the strings `"all"`, `"extending"`, which means that we consider either all possible class fusions of `s` into `tbl2` or only those whose image does not lie in G'. Note that the table of the derived subgroup of G is needed because we want to express the decomposition of the permutation characters relative to G'.

The idea is that we know that n different permutation characters arise from subgroups isomorphic with S (with the additional property that the image of the embedding of S into G is not contained in G' if the last argument is `"extending"`), and that `candidates` is a set of possible permutation characters, of length n. If the possible fusions between the character tables `s` and `tbl2` lead to exactly the given n permutation characters then we have proved that they are in fact the permutation characters of G in question. In this case, `VerifyCandidates` prints information about the decomposition of the permutation characters. If none of `candidates` arises from the possible embeddings of S into G then the function prints that S does not occur. In all other cases, the function signals an error. (This will not happen in the calls to this function below).

```
gap> VerifyCandidates:= function( s, tbl, tbl2, candidates, admissible )
>   local fus, der, pi;
>
>   if tbl2 = 0 then
>     tbl2:= tbl;
>   fi;
>
>   # Compute the possible class fusions, and induce the trivial character.
>   fus:= PossibleClassFusions( s, tbl2 );
>   if admissible = "extending" then
>     der:= Set( GetFusionMap( tbl, tbl2 ) );
>     fus:= Filtered( fus, map -> not IsSubset( der, map ) );
>   fi;
>   pi:= Set( List( fus, map -> Induced( s, tbl2,
>            [ TrivialCharacter( s ) ], map )[1] ) );
>
>   # Compare the two lists.
>   if pi = SortedList( candidates ) then
>     Print( "G = ", Identifier( tbl2 ), ":  point stabilizer ",
>            Identifier( s ), ", ranks ",
>            List( pi, x -> Length( ConstituentsOfCharacter(x) ) ), "\n" );
>     if Size( tbl ) = Size( tbl2 ) then
>       Print( PermCharInfo( tbl, pi ).ATLAS, "\n" );
>     else
```

```
>         Print( PermCharInfoRelative( tbl, tbl2, pi ).ATLAS, "\n" );
>       fi;
>     elif IsEmpty( Intersection( pi, candidates ) ) then
>       Print( "G = ", Identifier( tbl2 ), ":  no ", Identifier( s ), "\n" );
>     else
>       Error( "problem with verify" );
>     fi;
>     end;;
```

Since in most cases the character tables of possible point stabilizers are contained in the GAP Character Table Library, the above function provides an easy test. Alternatively, we could compute *all* faithful possible permutation characters (not only the multiplicity-free ones) of the degree in question; if there are as many different such characters as are known to be induced from point stabilizers *and* if no other subgroups of this index exist then the characters are indeed permutation characters, and we can compare them with the multiplicity-free characters computed before.

In the verification of the candidates, the following situations occur.

Lemma 2.1 *Let $\Phi: \hat{G} \to G$ be a group epimorphism, with $K = \ker(\Phi)$ cyclic of order m, and let H be a subgroup of G such that m is coprime to the order of the commutator factor group of H. Assume that it is known that $\Phi^{-1}(H)$ is a direct product of H with K. (This holds for example if H is simple and the order of the Schur multiplier of H is coprime to m.) Then the preimages under Φ of the G-conjugates of H contain one $\hat{G}$-class of subgroups that are isomorphic with H and that intersect trivially with K.*

Lemma 2.2 *Let $\Phi: \hat{G} \to G$ be a group epimorphism, with $K = \ker(\Phi)$ of order 3, such that the derived subgroup G' of G has index 2 in G and such that K is not central in $\hat{G}$. (So $\Phi^{-1}(G')$ is the centralizer of K in $\hat{G}$.) Consider a subgroup H of G with a subgroup $H_0 = H \cap G'$ of index 2 in H, and assume that the preimage $\Phi^{-1}(H_0)$ is a direct product of H_0 with K. (This holds for example if H_0 is simple and the order of the Schur multiplier of H_0 is coprime to 3.) Then each complement of K in $\Phi^{-1}(H_0)$ extends in $\Phi^{-1}(H)$ to exactly three complements of K that are isomorphic with H and conjugate in $\Phi^{-1}(H)$.*

Lemma 2.3 *Let $\Phi: \hat{G} \to G$ be a group epimorphism, with $K = \ker(\Phi)$ of order 2. Consider a subgroup H of G, with derived subgroup H' of index 2 in H and such that $\Phi^{-1}(H')$ is a direct product $K \times H'$.*

(i) Suppose that there is an element $h \in H \setminus H'$ such that the squares of the preimages of h in $\hat{G}$ lie in the unique subgroup of index 2 in $\Phi^{-1}(H')$. (This holds for example if the preimages of h are involutions.) Then $\Phi^{-1}(H)$ has the type $K \times H$.

(ii) If $\Phi^{-1}(H)$ has the type $K \times H$ then this group contains exactly two subgroups that are isomorphic with H. If H is a maximal subgroup of G then these two subgroups are not conjugate in $\hat{G}$.

(iii) Suppose that case (ii) applies and that there is $h \in H \setminus H'$ whose two preimages under Φ are not conjugate in $\hat{G}$ and such that each of the two subgroups of the type H in $\Phi^{-1}(H)$ contains elements in only one conjugacy class of $\hat{G}$ that contain the preimages of h. Then the two subgroups of the type H induce different permutation characters of $\hat{G}$, in particular exactly two conjugacy classes of subgroups of the type H in $\hat{G}$ arise from the conjugates of H in G.

With character theoretic methods, we can check a weaker form of Lemma 2.3 (i). Namely, the conditions are clearly satisfied if there is a conjugacy class C of elements in H that is not contained in H' and such that the class of $\hat{G}$ that contains the squares of the preimages of C is *not* contained in the images of the classes of $2 \times H'$ that lie outside H'.

The function `CheckConditionsForLemma3` tests this, and prints a message if Lemma 2.3 (i) applies because of this situation. More precisely, the arguments are (in this order) the character tables of

H', H, G, $\hat{G}$, and one of the strings "all", "extending"; the last argument expresses that either all embeddings of H into G are considered or only those which do not lie inside the derived subgroup of G.

The function *assumes* that s0 is the character table of the derived subgroup of the group of s, and that H' lifts to a direct product in $\hat{G}$.

```
gap> CheckConditionsForLemma3:= function( s0, s, fact, tbl, admissible )
>    local s0fuss, poss, der, sfusfact, outerins, outerinfact, preim,
>          squares, dp,  dpfustbl, s0indp, other, goodclasses;
>
>    if Size( s ) <> 2 * Size( s0 ) then
>      Error( "<s> must be twice as large as <s0>" );
>    fi;
>
>    s0fuss:= GetFusionMap( s0, s );
>    if s0fuss = fail then
>      poss:= Set( List( PossiblePermutationCharacters( s0, s ),
>                        pi -> Filtered( [ 1 .. Length( pi ) ],
>                                        i -> pi[i] <> 0 ) ) );
>      if Length( poss ) = 1 then
>        s0fuss:= poss[1];
>      else
>        Error( "classes of <s0> in <s> not determined" );
>      fi;
>    fi;
>    sfusfact:= PossibleClassFusions( s, fact );
>    if admissible = "extending" then
>      der:= ClassPositionsOfDerivedSubgroup( fact );
>      sfusfact:= Filtered( sfusfact, map -> not IsSubset( der, map ) );
>    fi;
>    outerins:= Difference( [ 1 .. NrConjugacyClasses( s ) ], s0fuss );
>    outerinfact:= Set( List( sfusfact, map -> Set( map{ outerins } ) ) );
>    if Length( outerinfact ) <> 1 then
>      Error( "classes of `", s, "' inside `", fact, "' not determined" );
>    fi;
>
>    preim:= Flat( InverseMap( GetFusionMap( tbl, fact ) ){ outerinfact[1] } );
>    squares:= Set( PowerMap( tbl, 2 ){ preim } );
>    dp:= s0 * CharacterTable( "Cyclic", 2 );
>    dpfustbl:= PossibleClassFusions( dp, tbl );
>    s0indp:= GetFusionMap( s0, dp );
>    other:= Difference( [ 1 .. NrConjugacyClasses( dp ) ], s0indp );
>    goodclasses:= List( dpfustbl, map -> Intersection( squares,
>                            Difference( map{ s0indp }, map{ other } ) ) );
>    if not IsEmpty( Intersection( goodclasses ) ) then
>      Print( Identifier( tbl ), ":  ", Identifier( s ),
>             " lifts to a direct product,\n",
>             "proved by squares in ", Intersection( goodclasses ), ".\n" );
>    elif ForAll( goodclasses, IsEmpty ) then
>      Print( Identifier( tbl ), ":  ", Identifier( s ),
>             " lifts to a nonsplit extension.\n" );
>    else
>      Print( "sorry, no proof of the splitting!\n" );
>    fi;
```

```
>    end;;
```

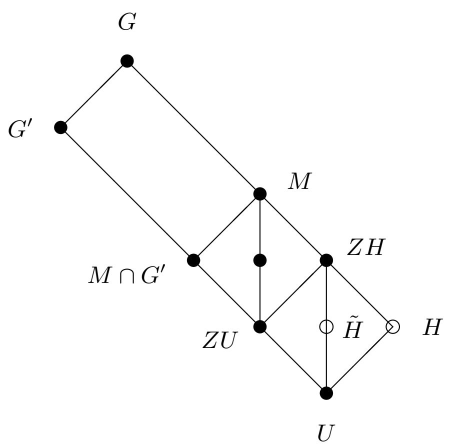
Lemma 2.3 (iii) can be utilized as follows. We assume the situation of Lemma 2.3, so $\Phi^{-1}(H)$ is a direct product $\langle z \rangle \times H$, where z is an involution. The derived subgroup of $\Phi^{-1}(H)$ is $H_0 \cong H'$, and $\Phi^{-1}(H)$ contains two subgroups H_1, H_2 which are isomorphic with H, and such that $H_2 = H_0 \cup \{hz; h \in H_1 \setminus H_0\}$. If the embedding of H_1, say, into $\hat{G}$ has the properties that an element outside H_0 is mapped into a class C of $\hat{G}$ that is different from zC and such that no element of H_1 lies in zC then zC contains elements of H_2 but C does not. In particular, the permutation characters of the two actions of $\hat{G}$ on the cosets of H_1 and H_2, respectively, are necessarily different.

We check this with the following function. Its arguments are one class fusion from the character table of H_1 to that of $\hat{G}$, the factor fusion from the character table of $\hat{G}$ to that of G, and the list of positions of the classes of H_0 in the character table of H_1. The return value is `true` if there are two different permutation characters, and `false` if this cannot be proved using the criterion.

```
gap> NecessarilyDifferentPermChars:= function( fusion, factfus, inner )
>    local outer, inv;
>
>    outer:= Difference( [ 1 .. Length( fusion ) ], inner );
>    fusion:= fusion{ outer };
>    inv:= Filtered( InverseMap( factfus ), IsList );
>    return ForAny( inv, pair -> Length( Intersection( pair, fusion ) ) = 1 );
>    end;;
```

The following observation is used to determine the number of conjugacy classes of certain subgroups.

Lemma 2.4 *Let G be a group with $[G : G'] = 2$, and $Z \subseteq Z(G) < G'$ with $|Z| = 2$. Consider a maximal subgroup M of G with $Z < M$ and $M \not\subseteq G'$, and a subgroup $H < M$ with $[M : H] = 4$ such that $U = H \cap G'$ is normal in M, $U \neq H$ holds, and $Z \not\subseteq H$. Let $N = ZH$. Then the three subgroups of index two in N that lie above U are ZU, H, and a group $\tilde{H}$, say. If M/U is a dihedral group of order eight then the groups H and $\tilde{H}$ are conjugate in M, and M/U is a dihedral group of order eight if and only if $M \setminus H$ contains both elements whose squares lie in U and elements whose squares do not lie in U.*



We want to detect that M/U is a dihedral group by character theoretic means but *without* using the character table of M. A sufficient (but not necessary) condition is that the set $D = \{g \in G \mid 1_M^G \neq 0, 1_N^G(g) = 0\}$ is nonempty and that there are elements g_1, $g_2 \in D$ with the properties $1_U^G(g_1^2) = 0$ and $|g_2| = 2$.

The following function takes the character table of G and the three permutation characters 1_U^G, 1_M^G, 1_N^G, and returns a list of length two, the i-th entry being the list of class positions of elements that can serve as g_i. So M/U is proved to be a dihedral group if both entries are nonempty.

```
gap> ProofOfD8Factor:= function( tblG, piU, piM, piN )
>    local D, map, D1, D2;
>
>    D:= Filtered( [ 1 .. Length( piU ) ], i -> piM[i] <> 0 and piN[i] = 0 );
>    map:= PowerMap( tblG, 2 );
>    D1:= Filtered( D, i -> piU[ map[i] ] = 0 );
>    D2:= Filtered( D, i -> OrdersClassRepresentatives( tblG )[i] = 2 );
>    return [ D1, D2 ];
>    end;;
```

2.3 Isoclinic Groups

For dealing with the character tables of groups of the type $2.G.2$ that are isoclinic to those whose tables are printed in the Atlas ([CCN+85]), it is necessary to store explicitly the factor fusion from $2.G.2$ onto $G.2$ and the subgroup fusion from $2.G$ into $2.G.2$, in order to make the above functions work. Note that these maps coincide for the two isoclinism types.

```
gap> IsoclinicTable:= function( tbl, tbl2, facttbl )
>    local subfus, factfus;
>
>    subfus:= GetFusionMap( tbl, tbl2 );
>    factfus:= GetFusionMap( tbl2, facttbl );
>    tbl2:= CharacterTableIsoclinic( tbl2 );
>    StoreFusion( tbl, subfus, tbl2 );
>    StoreFusion( tbl2, factfus, facttbl );
>    return tbl2;
>    end;;
```

2.4 Tests for GAP

With the following function, we check whether the characters computed here coincide with the characters stored in the data file.

```
gap> CompareWithDatabase:= function( name, chars )
>    local info;
>
>    info:= MultFreeEndoRingCharacterTables( name );
>    info:= List( info, x -> x.character );;
>    if SortedList( info ) <> SortedList( Concatenation( chars ) ) then
>      Error( "contradiction 1 for ", name );
>    fi;
>    end;;
```

If the character tables of all maximal subgroups of G are known then we could use alternatively the same method (and in fact the same GAP functions) as in the classification in [BL96]. This is shown in the following sections where applicable, using the following function. (The function `PossiblePermutationCharactersWithBoundedMultiplicity` is defined in the file `tst/multfree.g` of the GAP Character Table Library [Bre12]; note that it returns not only faithful characters.)

```
gap> CompareWithCandidatesByMaxes:= function( name, faith )
>    local tbl, poss;
>
>    tbl:= CharacterTable( name );
>    if not HasMaxes( tbl ) then
>      Error( "no maxes stored for ", name );
>    fi;
>    poss:= PossiblePermutationCharactersWithBoundedMultiplicity( tbl, 1 );
>    poss:= List( poss.permcand, l -> Filtered( l,
>                 pi -> ClassPositionsOfKernel( pi ) = [ 1 ] ) );
>    if SortedList( Concatenation( poss ) )
>       <> SortedList( Concatenation( faith ) ) then
>      Error( "contradiction 2 for ", name );
>    fi;
>    end;;
```

3 The Groups

In the following, we use ATLAS notation (see [CCN+85]) for the names of the groups. In particular, $2 \times G$ and $G \times 2$ denote the direct product of the group G with a cyclic group of order 2, and $G.2$ and $2.G$ denote an upward and downward extension, respectively, of G by a cyclic group of order 2, such that these groups are *not* direct products.

For groups of the structure $2.G.2$ where the character table of G is contained in the ATLAS, we use the name $2.G.2$ for the isoclinism type whose character table is printed in the ATLAS, and $(2.G.2)^*$ for the other isoclinism type.

Most of the computations that are shown in the following use only information from the GAP Character Table Library. The (few) explicit computations with groups are collected in Section 4.

3.1 $G = 2.M_{12}$

The group $2.M_{12}$ has ten faithful multiplicity-free permutation actions, with point stabilizers of the types M_{11} (twice), $A_6.2_1$ (twice), $3^2.2.S_4$ (four classes), and $3^2 : 2.A_4$ (twice).

```
gap> tbl:= CharacterTable( "2.M12" );;
gap> faith:= FaithfulCandidates( tbl, "M12" );;
1:  subgroup $M_{11}$, degree 24 (1 cand.)
2:  subgroup $M_{11}$, degree 24 (1 cand.)
5:  subgroup $A_6.2_1 \leq A_6.2^2$, degree 264 (1 cand.)
8:  subgroup $A_6.2_1 \leq A_6.2^2$, degree 264 (1 cand.)
11:  subgroup $3^2.2.S_4$, degree 440 (2 cand.)
12:  subgroup $3^2:2.A_4 \leq 3^2.2.S_4$, degree 880 (1 cand.)
13:  subgroup $3^2.2.S_4$, degree 440 (2 cand.)
14:  subgroup $3^2:2.A_4 \leq 3^2.2.S_4$, degree 880 (1 cand.)
```

There are two classes of M_{11} subgroups in M_{12} as well as in $2.M_{12}$, so we apply Lemma 2.1.

```
gap> VerifyCandidates( CharacterTable( "M11" ), tbl, 0,
>        Concatenation( faith[1], faith[2] ), "all" );
G = 2.M12:  point stabilizer M11, ranks [ 3, 3 ]
[ "1a+11a+12a", "1a+11b+12a" ]
```

According to the list of maximal subgroups of $2.M_{12}$, any $A_6.2^2$ subgroup in M_{12} lifts to a group of the structure $A_6.D_8$ in M_{12}, which contains two conjugate subgroups of the type $A_6.2_1$; so we get two classes of such subgroups, with the same permutation character.

```
gap> Maxes( tbl );
[ "2xM11", "2.M12M2", "A6.D8", "2.M12M4", "2.L2(11)", "2x3^2.2.S4",
  "2.M12M7", "2.M12M8", "2.M12M9", "2.M12M10", "2.A4xS3" ]
gap> faith[5] = faith[8];
true
gap> VerifyCandidates( CharacterTable( "A6.2_1" ), tbl, 0, faith[5], "all" );
G = 2.M12:  point stabilizer A6.2_1, ranks [ 7 ]
[ "1a+11ab+12a+54a+55a+120b" ]
```

The $3^2.2.S_4$ type subgroups of M_{12} lift to direct products with the centre of $2.M_{12}$, each such group contains two subgroups of the type $3^2.2.S_4$ which induce different permutation characters, for example because the involutions in $3^2.2.S_4 \setminus 3^2.2.A_4$ lie in the two preimages of the class 2B of M_{12}.

```
gap> s:= CharacterTable( "3^2.2.S4" );;
gap> derpos:= ClassPositionsOfDerivedSubgroup( s );;
gap> facttbl:= CharacterTable( "M12" );;
gap> factfus:= GetFusionMap( tbl, facttbl );;
gap> ForAll( PossibleClassFusions( s, tbl ),
>        map -> NecessarilyDifferentPermChars( map, factfus, derpos ) );
true
gap> VerifyCandidates( s, tbl, 0, Concatenation( faith[11], faith[13] ), "all" );
G = 2.M12:  point stabilizer 3^2.2.S4, ranks [ 7, 7, 9, 9 ]
[ "1a+11a+54a+55a+99a+110ab", "1a+11b+54a+55a+99a+110ab",
  "1a+11a+12a+44ab+54a+55a+99a+120b", "1a+11b+12a+44ab+54a+55a+99a+120b" ]
```

Each $3^2.2.S_4$ type group contains a unique subgroup of the type $3^2.2.A_4$, we get two classes of such subgroups, with different permutation characters because already the corresponding characters for M_{12} are different; we verify the candidates by inducing the degree two permutation characters of the $3^2.2.S_4$ type groups to $2.M_{12}$.

```
gap> fus:= PossibleClassFusions( s, tbl );;
gap> deg2:= PermChars( s, 2 );
[ Character( CharacterTable( "3^2.2.S4" ), [ 2, 2, 2, 2, 2, 2, 2, 0, 0, 0, 0
     ] ) ]
gap> pi:= Set( List( fus, map -> Induced( s, tbl, deg2, map )[1] ) );;
gap> pi = SortedList( Concatenation( faith[12], faith[14] ) );
true
gap> PermCharInfo( tbl, pi ).ATLAS;
[ "1a+11a+12a+44ab+45a+54a+55ac+99a+110ab+120ab",
  "1a+11b+12a+44ab+45a+54a+55ab+99a+110ab+120ab" ]
gap> CompareWithDatabase( "2.M12", faith );
gap> CompareWithCandidatesByMaxes( "2.M12", faith );
```

3.2 $G = 2.M_{12}.2$

The group $2.M_{12}.2$ that is printed in the ATLAS has three faithful multiplicity-free permutation actions, with point stabilizers of the types M_{11} and $L_2(11).2$ (twice), respectively.

```
gap> tbl2:= CharacterTable( "2.M12.2" );;
gap> faith:= FaithfulCandidates( tbl2, "M12.2" );;
1:  subgroup $M_{11}$, degree 48 (1 cand.)
2:  subgroup $L_2(11).2$, degree 288 (2 cand.)
```

The two classes of subgroups of the type M_{11} in $2.M_{12}$ are fused in $2.M_{12}.2$, so we get one class of these subgroups.

```
gap> VerifyCandidates( CharacterTable( "M11" ), tbl, tbl2, faith[1], "all" );
G = 2.M12.2:  point stabilizer M11, ranks [ 5 ]
[ "1a^{\\pm}+11ab+12a^{\\pm}" ]
```

The outer involutions in the maximal subgroups of the type $L_2(11).2$ in $M_{12}.2$ lift to involutions in $2.M_{12}.2$; moreover, those subgroups of the type $L_2(11).2$ that are novelties (so the intersection with M_{12} lies in M_{11} type subgroups) contain `2B` elements, which lift to involutions in $2.M_{12}.2$, so the $L_2(11)$ subgroup lifts to a group of the type $2 \times L_2(11)$, and Lemma 2.3 (ii) yields two classes of subgroups. The permutation characters are different, for example because each of the two candidates contains elements in one of the two preimages of the class `2B`.

(The function `CheckConditionsForLemma3` fails here, because of the two classes of maximal subgroups $L_2(11).2$ in $M_{12}.2$. One of them contains `2A` elements, the other contains `2B` elements. Only the latter type of subgroups, whose intersection with M_{12} is not maximal in M_{12}, lifts to subgroups of $2.M_{12}.2$ that contain $L_2(11).2$ subgroups.)

```
gap> s:= CharacterTable( "L2(11).2" );;
gap> derpos:= ClassPositionsOfDerivedSubgroup( s );;
gap> facttbl:= CharacterTable( "M12.2" );;
gap> factfus:= GetFusionMap( tbl2, facttbl );;
gap> ForAll( PossibleClassFusions( s, tbl2 ),
>        map -> NecessarilyDifferentPermChars( map, factfus, derpos ) );
true
gap> VerifyCandidates( s, tbl, tbl2, faith[2], "all" );
G = 2.M12.2:  point stabilizer L2(11).2, ranks [ 7, 7 ]
[ "1a^++11ab+12a^{\\pm}+55a^++66a^++120b^-",
  "1a^++11ab+12a^{\\pm}+55a^++66a^++120b^+" ]
gap> CompareWithDatabase( "2.M12.2", faith );
```

The group $(2.M_{12}.2)^*$ of the isoclinism type that is not printed in the ATLAS has one faithful multiplicity-free permutation action, with point stabilizer of the type M_{11}; as this subgroup lies inside $2.M_{12}$, its existence is clear, and the permutation character in both groups of the type $2.M_{12}.2$ is the same.

```
gap> tbl2:= IsoclinicTable( tbl, tbl2, facttbl );;
gap> faith:= FaithfulCandidates( tbl2, "M12.2" );;
1:  subgroup $M_{11}$, degree 48 (1 cand.)
gap> CompareWithDatabase( "Isoclinic(2.M12.2)", faith );
```

Note that in $(2.M_{12}.2)^*$, the subgroup of the type $(2 \times L_2(11)).2$ is a nonsplit extension, so the unique index 2 subgroup in this group contains the centre of $2.M_{12}.2$, in particular there is no subgroup of the type $L_2(11).2$.

```
gap> PossibleClassFusions( CharacterTable( "L2(11).2" ), tbl2 );
[  ]
```

3.3 $G = 2.M_{22}$

The group $2.M_{22}$ has four faithful multiplicity-free permutation actions, with point stabilizers of the types $2^4 : A_5$, A_7 (twice), and $2^3 : L_3(2)$, by Lemma 2.1.

```
gap> tbl:= CharacterTable( "2.M22" );;
gap> faith:= FaithfulCandidates( tbl, "M22" );;
3:  subgroup $2^4:A_5 \leq 2^4:A_6$, degree 924 (1 cand.)
4:  subgroup $A_7$, degree 352 (1 cand.)
5:  subgroup $A_7$, degree 352 (1 cand.)
7:  subgroup $2^3:L_3(2)$, degree 660 (1 cand.)
```

Note that one class of subgroups of the type $2^4 : A_5$ in the maximal subgroup of the type $2^4 : A_6$ as well as the A_7 and $2^3 : L_3(2)$ subgroups lift to direct products in $2.M_{22}$. A proof for $2^4 : A_5$ using explicit computations with the group can be found in Section 4.1.

```
gap> Maxes( tbl );
[ "2.L3(4)", "2.M22M2", "2xA7", "2xA7", "2.M22M5", "2x2^3:L3(2)",
  "(2xA6).2_3", "2xL2(11)" ]
gap> s:= CharacterTable( "P1/G1/L1/V1/ext2" );;
gap> VerifyCandidates( s, tbl, 0, faith[3], "all" );
G = 2.M22:  point stabilizer P1/G1/L1/V1/ext2, ranks [ 8 ]
[ "1a+21a+55a+126ab+154a+210b+231a" ]
gap> faith[4] = faith[5];
true
gap> VerifyCandidates( CharacterTable( "A7" ), tbl, 0, faith[4], "all" );
G = 2.M22:  point stabilizer A7, ranks [ 5 ]
[ "1a+21a+56a+120a+154a" ]
gap> VerifyCandidates( CharacterTable( "M22M6" ), tbl, 0, faith[7], "all" );
G = 2.M22:  point stabilizer 2^3:sl(3,2), ranks [ 7 ]
[ "1a+21a+55a+99a+120a+154a+210b" ]
gap> CompareWithDatabase( "2.M22", faith );
gap> CompareWithCandidatesByMaxes( "2.M22", faith );
```

3.4 $G = 2.M_{22}.2$

The group $2.M_{22}.2$ that is printed in the Atlas has eight faithful multiplicity-free permutation actions, with point stabilizers of the types $2^4 : S_5$ (twice), A_7, $2^3 : L_3(2) \times 2$ (twice), $2^3 : L_3(2)$, and $L_2(11).2$ (twice).

```
gap> tbl2:= CharacterTable( "2.M22.2" );;
gap> faith:= FaithfulCandidates( tbl2, "M22.2" );;
6:  subgroup $2^4:S_5 \leq 2^4:S_6$, degree 924 (2 cand.)
7:  subgroup $A_7$, degree 704 (1 cand.)
11:  subgroup $2^3:L_3(2) \times 2$, degree 660 (2 cand.)
12:  subgroup $2^3:L_3(2) \leq 2^3:L_3(2) \times 2$, degree 1320 (2 cand.)
16:  subgroup $L_2(11).2$, degree 1344 (2 cand.)
```

The character table of the $2^4 : S_5$ type subgroup is contained in the GAP Character Table Library, with identifier `w(d5)` (which denotes the Weyl group of the type D_5, cf. Section 4.2).

```
gap> s:= CharacterTable( "w(d5)" );;
gap> derpos:= ClassPositionsOfDerivedSubgroup( s );;
gap> facttbl:= CharacterTable( "M22.2" );;
gap> factfus:= GetFusionMap( tbl2, facttbl );;
gap> ForAll( PossibleClassFusions( s, tbl2 ),
>        map -> NecessarilyDifferentPermChars( map, factfus, derpos ) );
true
gap> VerifyCandidates( s, tbl, tbl2, faith[6], "all" );
```

```
G = 2.M22.2:  point stabilizer w(d5), ranks [ 7, 7 ]
[ "1a^++21a^++55a^++126ab+154a^++210b^-+231a^-",
  "1a^++21a^++55a^++126ab+154a^++210b^++231a^-" ]
```

The two classes of the type A_7 subgroups in $2.M_{22}$ are fused in $2.M_{22}.2$.

```
gap> VerifyCandidates( CharacterTable( "A7" ), tbl, tbl2, faith[7], "all" );
G = 2.M22.2:  point stabilizer A7, ranks [ 10 ]
[ "1a^{\\pm}+21a^{\\pm}+56a^{\\pm}+120a^{\\pm}+154a^{\\pm}" ]
```

The preimages of the $2^3 : L_3(2) \times 2$ type subgroups of $M_{22}.2$ in $2.M_{22}.2$ are direct products, by the discussion of $2.M_{22}$ and Lemma 2.3 (i). So Lemma 2.3 (iii) yields two classes, with different permutation characters.

```
gap> s:= CharacterTable( "2x2^3:L3(2)" );;
gap> s0:= CharacterTable( "2^3:sl(3,2)" );;
gap> CheckConditionsForLemma3( s0, s, facttbl, tbl2, "extending" );
2.M22.2:  2x2^3:L3(2) lifts to a direct product,
proved by squares in [ 1, 5, 14, 16 ].
gap> derpos:= ClassPositionsOfDerivedSubgroup( s );;
gap> ForAll( PossibleClassFusions( s, tbl2 ),
>        map -> NecessarilyDifferentPermChars( map, factfus, derpos ) );
true
gap> VerifyCandidates( s, tbl, tbl2, faith[11], "extending" );
G = 2.M22.2:  point stabilizer 2x2^3:L3(2), ranks [ 7, 7 ]
[ "1a^++21a^++55a^++99a^++120a^-+154a^++210b^-",
  "1a^++21a^++55a^++99a^++120a^++154a^++210b^+" ]
```

There is one class of subgroups of the type $2^3 : L_3(2)$ in $2.M_{22}$. One of the two candidates of degree 1 320 is excluded because it does not arise from a possible class fusion.

```
gap> s:= CharacterTable( "M22M6" );;
gap> pi1320:= PossiblePermutationCharacters( s, tbl2 );;
gap> Length( pi1320 );
1
gap> IsSubset( faith[12], pi1320 );
true
gap> faith[12]:= pi1320;;
gap> VerifyCandidates( s, tbl, tbl2, faith[12], "all" );
G = 2.M22.2:  point stabilizer 2^3:sl(3,2), ranks [ 14 ]
[ "1a^{\\pm}+21a^{\\pm}+55a^{\\pm}+99a^{\\pm}+120a^{\\pm}+154a^{\\pm}+210b^{\\\
pm}" ]
```

By Lemma 2.3 (i), the preimages of the $L_2(11).2$ type subgroups of $M_{22}.2$ in $2.M_{22}.2$ are direct products, so Lemma 2.3 (iii) yields two classes, with different permutation characters.

```
gap> s:= CharacterTable( "L2(11).2" );;
gap> s0:= CharacterTable( "L2(11)" );;
gap> CheckConditionsForLemma3( s0, s, facttbl, tbl2, "all" );
2.M22.2:  L2(11).2 lifts to a direct product,
proved by squares in [ 1, 4, 10, 13 ].
gap> derpos:= ClassPositionsOfDerivedSubgroup( s );;
gap> ForAll( PossibleClassFusions( s, tbl2 ),
>        map -> NecessarilyDifferentPermChars( map, factfus, derpos ) );
true
```

```
gap> VerifyCandidates( CharacterTable( "L2(11).2" ), tbl, tbl2, faith[16], "all" );
G = 2.M22.2:  point stabilizer L2(11).2, ranks [ 10, 10 ]
[ "1a^++21a^-+55a^++56a^{\\pm}+120a^-+154a^++210a^-+231a^-+440a^+",
  "1a^++21a^-+55a^++56a^{\\pm}+120a^++154a^++210a^-+231a^-+440a^-" ]
gap> CompareWithDatabase( "2.M22.2", faith );
```

The group $(2.M_{22}.2)^*$ of the isoclinism type that is not printed in the Atlas has two faithful multiplicity-free permutation actions, with point stabilizers of the types A_7 and $2^3 : L_3(2)$.

```
gap> tbl2:= IsoclinicTable( tbl, tbl2, facttbl );;
gap> faith:= FaithfulCandidates( tbl2, "M22.2" );;
7:  subgroup $A_7$, degree 704 (1 cand.)
12:  subgroup $2^3:L_3(2) \leq 2^3:L_3(2) \times 2$, degree 1320 (2 cand.)
gap> faith[12]:= Filtered( faith[12], chi -> chi in pi1320 );;
gap> CompareWithDatabase( "Isoclinic(2.M22.2)", faith );
```

The two classes of subgroups lie inside $2.M_{22}$, so their existence has been discussed already above.

3.5 $G = 3.M_{22}$

The group $3.M_{22}$ has four faithful multiplicity-free permutation actions, with point stabilizers of the types $2^4 : A_5$, $2^4 : S_5$, $2^3 : L_3(2)$, and $L_2(11)$.

```
gap> tbl:= CharacterTable( "3.M22" );;
gap> faith:= FaithfulCandidates( tbl, "M22" );;
3:  subgroup $2^4:A_5 \leq 2^4:A_6$, degree 1386 (1 cand.)
6:  subgroup $2^4:S_5$, degree 693 (1 cand.)
7:  subgroup $2^3:L_3(2)$, degree 990 (1 cand.)
9:  subgroup $L_2(11)$, degree 2016 (1 cand.)
```

The existence of one class of each of these types follows from Lemma 2.1.

```
gap> VerifyCandidates( CharacterTable( "P1/G1/L1/V1/ext2" ), tbl, 0, faith[3], "all" );
G = 3.M22:  point stabilizer P1/G1/L1/V1/ext2, ranks [ 13 ]
[ "1a+21abc+55a+105abcd+154a+231abc" ]
gap> VerifyCandidates( CharacterTable( "M22M5" ), tbl, 0, faith[6], "all" );
G = 3.M22:  point stabilizer 2^4:s5, ranks [ 10 ]
[ "1a+21abc+55a+105abcd+154a" ]
gap> VerifyCandidates( CharacterTable( "M22M6" ), tbl, 0, faith[7], "all" );
G = 3.M22:  point stabilizer 2^3:sl(3,2), ranks [ 13 ]
[ "1a+21abc+55a+99abc+105abcd+154a" ]
gap> VerifyCandidates( CharacterTable( "M22M8" ), tbl, 0, faith[9], "all" );
G = 3.M22:  point stabilizer L2(11), ranks [ 16 ]
[ "1a+21abc+55a+105abcd+154a+210abc+231abc" ]
gap> CompareWithDatabase( "3.M22", faith );
gap> CompareWithCandidatesByMaxes( "3.M22", faith );
```

3.6 $G = 3.M_{22}.2$

The group $3.M_{22}.2$ has five faithful multiplicity-free permutation actions, with point stabilizers of the types $2^4 : S_5$, $2^5 : S_5$, $2^4 : (A_5 \times 2)$, $2^3 : L_3(2) \times 2$, and $L_2(11).2$.

```
gap> tbl2:= CharacterTable( "3.M22.2" );;
gap> faith:= FaithfulCandidates( tbl2, "M22.2" );;
6:  subgroup $2^4:S_5 \leq 2^4:S_6$, degree 1386 (1 cand.)
8:  subgroup $2^5:S_5$, degree 693 (1 cand.)
10:  subgroup $2^4:(A_5 \times 2) \leq 2^5:S_5$, degree 1386 (1 cand.)
11:  subgroup $2^3:L_3(2) \times 2$, degree 990 (1 cand.)
16:  subgroup $L_2(11).2$, degree 2016 (1 cand.)
```

Subgroups of these types exist by Lemma 2.2. The verification is straightforward in all cases except that of $2^4 : (A_5 \times 2)$.

```
gap> VerifyCandidates( CharacterTable( "w(d5)" ), tbl, tbl2, faith[6], "all" );
G = 3.M22.2:  point stabilizer w(d5), ranks [ 9 ]
[ "1a^++21a^+bc+55a^++105abcd+154a^++231a^-bc" ]
gap> VerifyCandidates( CharacterTable( "M22.2M4" ), tbl, tbl2, faith[8], "all" );
G = 3.M22.2:  point stabilizer M22.2M4, ranks [ 7 ]
[ "1a^++21a^+bc+55a^++105abcd+154a^+" ]
gap> VerifyCandidates( CharacterTable( "2x2^3:L3(2)" ), tbl, tbl2, faith[11], "all" );
G = 3.M22.2:  point stabilizer 2x2^3:L3(2), ranks [ 9 ]
[ "1a^++21a^+bc+55a^++99a^+bc+105abcd+154a^+" ]
gap> VerifyCandidates( CharacterTable( "L2(11).2" ), tbl, tbl2, faith[16], "all" );
G = 3.M22.2:  point stabilizer L2(11).2, ranks [ 11 ]
[ "1a^++21a^-bc+55a^++105abcd+154a^++210a^-bc+231a^-bc" ]
```

In the remaining case, we note that the $2^4 : (A_5 \times 2)$ type subgroup has index 2 in the maximal subgroup of the type $2^5 : S_5$, whose character table is available via the identifier M22.2M4. It is sufficient to show that exactly one of the three index 2 subgroups in this group induces a multiplicity-free permutation character of $3.M_{22}.2$, and this can be done by inducing the degree 2 permutation characters of $2^5 : S_5$ to $3.M_{22}.2$.

```
gap> s:= CharacterTable( "M22.2M4" );;
gap> lin:= LinearCharacters( s );
[ Character( CharacterTable( "M22.2M4" ), [ 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
      1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 ] ),
  Character( CharacterTable( "M22.2M4" ), [ 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
      1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1 ] ),
  Character( CharacterTable( "M22.2M4" ), [ 1, 1, 1, 1, 1, 1, 1, -1, -1, -1,
      -1, -1, 1, 1, 1, 1, 1, 1, 1, -1, -1, -1, -1, -1 ] ),
  Character( CharacterTable( "M22.2M4" ), [ 1, 1, 1, 1, 1, 1, 1, -1, -1, -1,
      -1, -1, -1, -1, -1, -1, -1, -1, -1, 1, 1, 1, 1, 1 ] ) ]
gap> perms:= List( lin{ [ 2 .. 4 ] }, chi -> chi + lin[1] );;
gap> sfustbl2:= PossibleClassFusions( s, tbl2 );;
gap> Length( sfustbl2 );
2
gap> ind1:= Induced( s, tbl2, perms, sfustbl2[1] );;
gap> ind2:= Induced( s, tbl2, perms, sfustbl2[2] );;
gap> PermCharInfo( tbl2, ind1 ).ATLAS;
[ "1ab+21ab+42aa+55ab+154ab+210ccdd", "1a+21ab+42a+55a+154a+210bcd+462a",
  "1a+21aa+42a+55a+154a+210acd+462a" ]
gap> PermCharInfo( tbl2, ind2 ).ATLAS;
[ "1a+21aa+42a+55a+154a+210acd+462a", "1a+21ab+42a+55a+154a+210bcd+462a",
  "1ab+21ab+42aa+55ab+154ab+210ccdd" ]
gap> ind1[2] = ind2[2];
true
gap> [ ind1[2] ] = faith[10];
```

```
true
gap> CompareWithDatabase( "3.M22.2", faith );
```

3.7 $G = 4.M_{22}$ and $G = 12.M_{22}$

The group $4.M_{22}$ and hence also the group $12.M_{22}$ has no faithful multiplicity-free permutation action.

```
gap> tbl:= CharacterTable( "4.M22" );;
gap> faith:= FaithfulCandidates( tbl, "2.M22" );;
gap> CompareWithDatabase( "4.M22", faith );
gap> CompareWithCandidatesByMaxes( "4.M22", faith );
```

3.8 $G = 4.M_{22}.2$ and $G = 12.M_{22}.2$

The two isoclinism types of groups of the type $4.M_{22}.2$ and hence also all groups of the type $12.M_{22}.2$ have no faithful multiplicity-free permutation actions.

```
gap> tbl2:= CharacterTable( "4.M22.2" );;
gap> faith:= FaithfulCandidates( tbl2, "M22.2" );;
gap> CompareWithDatabase( "4.M22.2", faith );
gap> CompareWithDatabase( "12.M22.2", [] );
gap> tbl2:= IsoclinicTable( tbl, tbl2, facttbl );;
gap> faith:= FaithfulCandidates( tbl2, "M22.2" );;
gap> CompareWithDatabase( "Isoclinic(4.M22.2)", faith );
gap> CompareWithDatabase( "Isoclinic(12.M22.2)", [] );
```

3.9 $G = 6.M_{22}$

The group $6.M_{22}$ has two faithful multiplicity-free permutation actions, with point stabilizers of the types $2^4 : A_5$ and $2^3 : L_3(2)$.

```
gap> tbl:= CharacterTable( "6.M22" );;
gap> faith:= FaithfulCandidates( tbl, "3.M22" );;
1:  subgroup $2^4:A_5 \rightarrow (M_{22},3)$, degree 2772 (1 cand.)
3:  subgroup $2^3:L_3(2) \rightarrow (M_{22},7)$, degree 1980 (1 cand.)
```

The existence of one class of each of these subgroups follows from the treatment of $2.M_{22}$ and $3.M_{22}$.

```
gap> VerifyCandidates( CharacterTable( "P1/G1/L1/V1/ext2" ), tbl, 0, faith[1], "all" );
G = 6.M22:  point stabilizer P1/G1/L1/V1/ext2, ranks [ 22 ]
[ "1a+21abc+55a+105abcd+126abcdef+154a+210bef+231abc" ]
gap> VerifyCandidates( CharacterTable( "M22M6" ), tbl, 0, faith[3], "all" );
G = 6.M22:  point stabilizer 2^3:sl(3,2), ranks [ 17 ]
[ "1a+21abc+55a+99abc+105abcd+120a+154a+210b+330de" ]
gap> CompareWithDatabase( "6.M22", faith );
gap> CompareWithCandidatesByMaxes( "6.M22", faith );
```

3.10 $G = 6.M_{22}.2$

The group $6.M_{22}.2$ that is printed in the ATLAS has six faithful multiplicity-free permutation actions, with point stabilizers of the types $2^4 : S_5$ (twice), $2^3 : L_3(2) \times 2$ (twice), and $L_2(11).2$ (twice).

```
gap> tbl2:= CharacterTable( "6.M22.2" );;
gap> faith:= FaithfulCandidates( tbl2, "M22.2" );;
6:  subgroup $2^4:S_5 \leq 2^4:S_6$, degree 2772 (2 cand.)
11:  subgroup $2^3:L_3(2) \times 2$, degree 1980 (2 cand.)
16:  subgroup $L_2(11).2$, degree 4032 (2 cand.)
```

We know that $2.M_{22}.2$ contains two classes of subgroups isomorphic with each of the required point stabilizers, so we apply Lemma 2.2.

```
gap> s:= CharacterTable( "w(d5)" );;
gap> VerifyCandidates( s, tbl, tbl2, faith[6], "all" );
G = 6.M22.2:  point stabilizer w(d5), ranks [ 14, 14 ]
[ "1a^++21a^+bc+55a^++105abcd+126abcdef+154a^++210b^-ef+231a^-bc",
  "1a^++21a^+bc+55a^++105abcd+126abcdef+154a^++210b^+ef+231a^-bc" ]
```

(Since $6.M_{22}$ contains subgroups of the type $2^3 : L_3(2) \times 2$ in which we are not interested, we must use "extending" as the last argument of VerifyCandidates for this case.)

```
gap> s:= CharacterTable( "2x2^3:L3(2)" );;
gap> VerifyCandidates( s, tbl, tbl2, faith[11], "extending" );
G = 6.M22.2:  point stabilizer 2x2^3:L3(2), ranks [ 12, 12 ]
[ "1a^++21a^+bc+55a^++99a^+bc+105abcd+120a^-+154a^++210b^-+330de",
  "1a^++21a^+bc+55a^++99a^+bc+105abcd+120a^++154a^++210b^++330de" ]
gap> VerifyCandidates( CharacterTable( "L2(11).2" ), tbl, tbl2, faith[16], "all" );
G = 6.M22.2:  point stabilizer L2(11).2, ranks [ 20, 20 ]
[ "1a^++21a^-bc+55a^++56a^{\\pm}+66abcd+105abcd+120a^-bc+154a^++210a^-cdghij+2\
31a^-bc+440a^+",
  "1a^++21a^-bc+55a^++56a^{\\pm}+66abcd+105abcd+120a^+bc+154a^++210a^-cdghij+2\
31a^-bc+440a^-" ]
gap> CompareWithDatabase( "6.M22.2", faith );
```

The group $(6.M_{22}.2)^*$ of the isoclinism type that is not printed in the Atlas has no faithful multiplicity-free permutation action.

```
gap> tbl2:= IsoclinicTable( tbl, tbl2, facttbl );;
gap> faith:= FaithfulCandidates( tbl2, "M22.2" );;
gap> CompareWithDatabase( "Isoclinic(6.M22.2)", faith );
```

3.11 $G = 2.J_2$

The group $2.J_2$ has one faithful multiplicity-free permutation action, with point stabilizer of the type $U_3(3)$, by Lemma 2.1.

```
gap> tbl:= CharacterTable( "2.J2" );;
gap> faith:= FaithfulCandidates( tbl, "J2" );;
1:  subgroup $U_3(3)$, degree 200 (1 cand.)
gap> VerifyCandidates( CharacterTable( "U3(3)" ), tbl, 0, faith[1], "all" );
G = 2.J2:  point stabilizer U3(3), ranks [ 5 ]
[ "1a+36a+50ab+63a" ]
gap> CompareWithDatabase( "2.J2", faith );
gap> CompareWithCandidatesByMaxes( "2.J2", faith );
```

3.12 $G = 2.J_2.2$

The group $2.J_2.2$ that is printed in the ATLAS has no faithful multiplicity-free permutation action.

```
gap> tbl2:= CharacterTable( "2.J2.2" );;
gap> faith:= FaithfulCandidates( tbl2, "J2.2" );;
gap> CompareWithDatabase( "2.J2.2", faith );
```

The group $(2.J_2.2)^*$ of the isoclinism type that is not printed in the ATLAS has three faithful multiplicity-free permutation actions, with point stabilizers of the types $U_3(3).2$ (twice) and $3.A_6.2_3$.

```
gap> facttbl:= CharacterTable( "J2.2" );;
gap> tbl2:= IsoclinicTable( tbl, tbl2, facttbl );;
gap> faith:= FaithfulCandidates( tbl2, "J2.2" );;
1:  subgroup $U_3(3).2$, degree 200 (1 cand.)
5:  subgroup $3.A_6.2_3 \leq 3.A_6.2^2$, degree 1120 (1 cand.)
```

The existence of two classes of $U_3(3)$ type subgroups follows from Lemma 2.3 (ii).

```
gap> s0:= CharacterTable( "U3(3)" );;
gap> s:= CharacterTable( "U3(3).2" );;
gap> CheckConditionsForLemma3( s0, s, facttbl, tbl2, "all" );
Isoclinic(2.J2.2):  U3(3).2 lifts to a direct product,
proved by squares in [ 1, 3, 8, 16 ].
gap> VerifyCandidates( s, tbl, tbl2, faith[1], "all" );
G = Isoclinic(2.J2.2):  point stabilizer U3(3).2, ranks [ 4 ]
[ "1a^++36a^++50ab+63a^+" ]
```

Each maximal subgroup of the type $3.A_6.2^2$ in $J_2.2$ contains a subgroup U of the type $3.A_6.2_3$, which lifts to a direct product $N = 2 \times 3.A_6.2_3$ in $(2.J_2.2)^*$.

```
gap> s0:= CharacterTable( "3.A6" );;
gap> s:= CharacterTable( "3.A6.2_3" );;
gap> CheckConditionsForLemma3( s0, s, facttbl, tbl2, "all" );
Isoclinic(2.J2.2):  3.A6.2_3 lifts to a direct product,
proved by squares in [ 3, 10, 16, 25 ].
```

There is only one class of $3.A_6.2_3$ type subgroups in each maximal subgroup M of $G = (2.J_2.2)^*$ that is a preimage of a $3.A_6.2^2$ type subgroup in $J_2.2$.

This follows from the fact that the normalizer of $H = 3.A_6.2_3$ in G is N; equivalently, the factor group of M modulo $U = H'$ is a dihedral group of order 8. With character-theoretic methods, this can be seen as follows.

```
gap> tblMbar:= CharacterTable( "3.A6.2^2" );;
gap> piMbar:= PossiblePermutationCharacters( tblMbar, facttbl );
[ Character( CharacterTable( "J2.2" ), [ 280, 40, 12, 1, 4, 4, 10, 0, 1, 0,
      0, 2, 2, 0, 1, 1, 14, 10, 0, 2, 4, 0, 1, 0, 0, 1, 1 ] ) ]
gap> piM:= piMbar[1]{ GetFusionMap( tbl2, facttbl ) };;
gap> piNbar:= PossiblePermutationCharacters( s, facttbl );
[ Character( CharacterTable( "J2.2" ), [ 560, 80, 0, 2, 8, 8, 20, 0, 2, 0, 0,
      0, 0, 0, 2, 2, 0, 8, 0, 0, 8, 0, 2, 0, 0, 2, 2 ] ) ]
gap> piN:= piNbar[1]{ GetFusionMap( tbl2, facttbl ) };;
gap> piU:= PossiblePermutationCharacters( s0, tbl2 );
[ Character( CharacterTable( "Isoclinic(2.J2.2)" ),
    [ 2240, 0, 320, 0, 0, 8, 0, 32, 0, 32, 0, 80, 0, 0, 0, 8, 0, 0, 0, 0, 0,
```

```
        0, 0, 0, 8, 0, 8, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
        0 ] ) ]
gap> ProofOfD8Factor( tbl2, piU[1], piM, piN );
[ [ 5, 21, 22 ], [ 29 ] ]
```

It remains to verify the candidate.

```
gap> VerifyCandidates( s, tbl, tbl2, faith[5], "all" );
G = Isoclinic(2.J2.2):  point stabilizer 3.A6.2_3, ranks [ 12 ]
[ "1a^++14c^{\\pm}+21ab+50ab+63a^{\\pm}+90a^++126a^++175a^-+216a^{\\pm}" ]
gap> faith[1]:= faith[1]{ [ 1, 1 ] };;
gap> CompareWithDatabase( "Isoclinic(2.J2.2)", faith );
```

3.13 $G = 2.HS$

The group $2.HS$ has five faithful multiplicity-free permutation actions, with point stabilizers of the types $U_3(5)$ (twice), A_8, and M_{11} (twice).

```
gap> tbl:= CharacterTable( "2.HS" );;
gap> faith:= FaithfulCandidates( tbl, "HS" );;
3:  subgroup $U_3(5) \leq U_3(5).2$, degree 704 (1 cand.)
5:  subgroup $U_3(5) \leq U_3(5).2$, degree 704 (1 cand.)
8:  subgroup $A_8 \leq A_8.2$, degree 4400 (1 cand.)
10:  subgroup $M_{11}$, degree 11200 (1 cand.)
11:  subgroup $M_{11}$, degree 11200 (1 cand.)
```

Lemma 2.1 applies in all cases; note that $2.HS$ does not admit an embedding of $2.A_8$.

```
gap> VerifyCandidates( CharacterTable( "U3(5)" ), tbl, 0,
>       Concatenation( faith[3], faith[5] ), "all" );
G = 2.HS:  point stabilizer U3(5), ranks [ 6, 6 ]
[ "1a+22a+154c+175a+176ab", "1a+22a+154b+175a+176ab" ]
gap> PossibleClassFusions( CharacterTable( "2.A8" ), tbl );
[  ]
gap> VerifyCandidates( CharacterTable( "A8" ), tbl, 0, faith[8], "all" );
G = 2.HS:  point stabilizer A8, ranks [ 13 ]
[ "1a+22a+77a+154abc+175a+176ab+693a+770a+924ab" ]
gap> VerifyCandidates( CharacterTable( "M11" ), tbl, 0,
>       Concatenation( faith[10], faith[11] ), "all" );
G = 2.HS:  point stabilizer M11, ranks [ 16, 16 ]
[ "1a+22a+56a+77a+154c+175a+176ab+616ab+770a+825a+1056a+1980ab+2520a",
  "1a+22a+56a+77a+154b+175a+176ab+616ab+770a+825a+1056a+1980ab+2520a" ]
gap> CompareWithDatabase( "2.HS", faith );
gap> CompareWithCandidatesByMaxes( "2.HS", faith );
```

3.14 $G = 2.HS.2$

The group $2.HS.2$ that is printed in the ATLAS has two faithful multiplicity-free permutation actions, with point stabilizers of the types $A_8 \times 2$ and $A_8.2$.

```
gap> tbl2:= CharacterTable( "2.HS.2" );;
gap> faith:= FaithfulCandidates( tbl2, "HS.2" );;
10:  subgroup $A_8 \times 2 \leq A_8.2 \times 2$, degree 4400 (1 cand.)
11:  subgroup $A_8.2 \leq A_8.2 \times 2$, degree 4400 (1 cand.)
```

The existence of subgroups for each candidate follows from Lemma 2.3. (Since there are $A_8 \times 2$ type subgroups inside $2.HS$ in which we are not interested, we must use `"extending"` as the last argument of `VerifyCandidates`.)

```
gap> facttbl:= CharacterTable( "HS.2" );;
gap> factfus:= GetFusionMap( tbl2, facttbl );;
gap> s0:= CharacterTable( "A8" );;
gap> s:= s0 * CharacterTable( "Cyclic", 2 );
CharacterTable( "A8xC2" )
gap> CheckConditionsForLemma3( s0, s, facttbl, tbl2, "all" );
2.HS.2:  A8xC2 lifts to a direct product,
proved by squares in [ 1, 6, 13, 20, 30 ].
gap> VerifyCandidates( s, tbl, tbl2, faith[10], "extending" );
G = 2.HS.2:  point stabilizer A8xC2, ranks [ 10 ]
[ "1a^++22a^++77a^++154a^+bc+175a^++176ab+693a^++770a^++924ab" ]
gap> s:= CharacterTable( "A8.2" );;
gap> CheckConditionsForLemma3( s0, s, facttbl, tbl2, "extending" );
2.HS.2:  A8.2 lifts to a direct product,
proved by squares in [ 1, 6, 13 ].
gap> VerifyCandidates( s, tbl, tbl2, faith[11], "all" );
G = 2.HS.2:  point stabilizer A8.2, ranks [ 10 ]
[ "1a^++22a^-+77a^++154a^+bc+175a^++176ab+693a^++770a^-+924ab" ]
gap> CompareWithDatabase( "2.HS.2", faith );
```

Note that any maximal $S_8 \times 2$ type subgroup in $HS.2$ contains two subgroups of the type S_8, and the one that is contained in HS does *not* lift to a direct product in $G = 2.HS.2$ but to a subdirect product S of S_8 and a cyclic group of order four, since $2.HS$ does not contain S_8 type subgroups.

Let M be a maximal subgroup of G that maps to a subgroup of the type $S_8 \times 2$ in the factor group $HS.2$. By the above observations, we know three subgroups of index two in M: the subdirect product S and the direct products $S_8 \times 2$ and $A_8 \times 2^2$. So we see that the factor group of M by the A_8 type subgroup is a dihedral group of order eight.

(The situation is similar to that in Section 3.12, but the sufficient condition checked by the function `ProofOfD8Factor` is not satisfied here, as the following computation shows. We have $U \cong A_8$ and $N \cong A_8 \times 2^2$.)

```
gap> tblMbar:= CharacterTable( "A8.2" ) * CharacterTable( "Cyclic", 2 );;
gap> piMbar:= PossiblePermutationCharacters( tblMbar, facttbl );
[ Character( CharacterTable( "HS.2" ), [ 1100, 60, 32, 11, 40, 16, 4, 0, 10,
      0, 5, 3, 1, 2, 0, 0, 2, 0, 1, 1, 0, 134, 30, 10, 10, 0, 11, 5, 3, 0, 4,
      4, 0, 1, 1, 0, 0, 0, 1 ] ) ]
gap> piM:= piMbar[1]{ GetFusionMap( tbl2, facttbl ) };;
gap> s:= s0 * CharacterTable( "Cyclic", 2 );;
gap> piNbar:= PossiblePermutationCharacters( s, facttbl );
[ Character( CharacterTable( "HS.2" ), [ 2200, 120, 0, 22, 0, 16, 8, 0, 20,
      0, 0, 6, 2, 0, 0, 0, 0, 0, 0, 2, 0, 212, 20, 20, 12, 0, 2, 8, 2, 0, 0,
      2, 0, 0, 2, 0, 0, 0, 2 ] ) ]
gap> piN:= piNbar[1]{ GetFusionMap( tbl2, facttbl ) };;
gap> piU:= PossiblePermutationCharacters( s0, tbl2 );
[ Character( CharacterTable( "2.HS.2" ), [ 8800, 0, 320, 160, 0, 88, 0, 0,
      32, 16, 0, 0, 80, 0, 0, 0, 0, 8, 16, 8, 0, 0, 0, 0, 0, 0, 0, 0, 0, 8,
      0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
      0, 0, 0 ] ) ]
gap> ProofOfD8Factor( tbl2, piU[1], piM, piN );
[ [ 5, 17, 26 ], [  ] ]
```

The group $(2.HS.2)^*$ of the isoclinism type that is not printed in the ATLAS has no faithful multiplicity-free permutation action.

```
gap> tbl2:= IsoclinicTable( tbl, tbl2, facttbl );;
gap> faith:= FaithfulCandidates( tbl2, "HS.2" );;
gap> CompareWithDatabase( "Isoclinic(2.HS.2)", faith );
```

3.15 $G = 3.J_3$

The group $3.J_3$ has no faithful multiplicity-free permutation action.

```
gap> tbl:= CharacterTable( "3.J3" );;
gap> faith:= FaithfulCandidates( tbl, "J3" );;
gap> CompareWithDatabase( "3.J3", faith );
```

3.16 $G = 3.J_3.2$

The group $3.J_3.2$ has no faithful multiplicity-free permutation action.

```
gap> tbl2:= CharacterTable( "3.J3.2" );;
gap> faith:= FaithfulCandidates( tbl2, "J3.2" );;
gap> CompareWithDatabase( "3.J3.2", faith );
```

3.17 $G = 3.McL$

The group $3.McL$ has one faithful multiplicity-free permutation action, with point stabilizer of the type $2.A_8$, by Lemma 2.1.

```
gap> tbl:= CharacterTable( "3.McL" );;
gap> faith:= FaithfulCandidates( tbl, "McL" );;
6:  subgroup $2.A_8$, degree 66825 (1 cand.)
gap> VerifyCandidates( CharacterTable( "2.A8" ), tbl, 0, faith[6], "all" );
G = 3.McL:  point stabilizer 2.A8, ranks [ 14 ]
[ "1a+252a+1750a+2772ab+5103abc+5544a+6336ab+8064ab+9625a" ]
gap> CompareWithDatabase( "3.McL", faith );
gap> CompareWithCandidatesByMaxes( "3.McL", faith );
```

3.18 $G = 3.McL.2$

The group $3.McL.2$ has one faithful multiplicity-free permutation action, with point stabilizer of the type $(2.A_8.2)^*$, by Lemma 2.2.

```
gap> tbl2:= CharacterTable( "3.McL.2" );;
gap> faith:= FaithfulCandidates( tbl2, "McL.2" );;
9:  subgroup $2.S_8$, degree 66825 (1 cand.)
gap> s:= CharacterTable( "Isoclinic(2.A8.2)" );;
gap> VerifyCandidates( s, tbl, tbl2, faith[9], "all" );
G = 3.McL.2:  point stabilizer Isoclinic(2.A8.2), ranks [ 10 ]
[ "1a^++252a^++1750a^++2772ab+5103a^+bc+5544a^++6336ab+8064ab+9625a^+" ]
gap> CompareWithDatabase( "3.McL.2", faith );
```

3.19 $G = 2.Ru$

The group $2.Ru$ has one faithful multiplicity-free permutation action, with point stabilizer of the type ${}^2F_4(2)'$, by Lemma 2.1.

```
gap> tbl:= CharacterTable( "2.Ru" );;
gap> faith:= FaithfulCandidates( tbl, "Ru" );;
2:  subgroup ${^2F_4(2)^{\prime}} \leq {^2F_4(2)^{\prime}}.2$, degree 16240 (
1 cand.)
gap> VerifyCandidates( CharacterTable( "2F4(2)'" ), tbl, 0, faith[2], "all" );
G = 2.Ru:  point stabilizer 2F4(2)', ranks [ 9 ]
[ "1a+28ab+406a+783a+3276a+3654a+4032ab" ]
gap> CompareWithDatabase( "2.Ru", faith );
```

3.20 $G = 2.Suz$

The group $2.Suz$ has one faithful multiplicity-free permutation action, with point stabilizer of the type $U_5(2)$, by Lemma 2.1.

```
gap> tbl:= CharacterTable( "2.Suz" );;
gap> faith:= FaithfulCandidates( tbl, "Suz" );;
4:  subgroup $U_5(2)$, degree 65520 (1 cand.)
gap> VerifyCandidates( CharacterTable( "U5(2)" ), tbl, 0, faith[4], "all" );
G = 2.Suz:  point stabilizer U5(2), ranks [ 10 ]
[ "1a+143a+364abc+5940a+12012a+14300a+16016ab" ]
gap> CompareWithDatabase( "2.Suz", faith );
```

3.21 $G = 2.Suz.2$

The group $2.Suz.2$ that is printed in the Atlas has four faithful multiplicity-free permutation actions, with point stabilizers of the types $U_5(2).2$ (twice) and $3^5 : (M_{11} \times 2)$ (twice), respectively.

```
gap> tbl2:= CharacterTable( "2.Suz.2" );;
gap> faith:= FaithfulCandidates( tbl2, "Suz.2" );;
8:  subgroup $U_5(2).2$, degree 65520 (1 cand.)
12:  subgroup $3^5:(M_{11} \times 2)$, degree 465920 (1 cand.)
```

We verify the conditions of Lemma 2.3 (ii).

```
gap> s0:= CharacterTable( "U5(2)" );;
gap> s:= CharacterTable( "U5(2).2" );;
gap> facttbl:= CharacterTable( "Suz.2" );;
gap> CheckConditionsForLemma3( s0, s, facttbl, tbl2, "all" );
2.Suz.2:  U5(2).2 lifts to a direct product,
proved by squares in [ 1, 8, 13, 19, 31, 44 ].
gap> VerifyCandidates( s, tbl, tbl2, faith[8], "all" );
G = 2.Suz.2:  point stabilizer U5(2).2, ranks [ 8 ]
[ "1a^++143a^-+364a^+bc+5940a^++12012a^-+14300a^-+16016ab" ]
gap> s0:= CharacterTable( "SuzM5" );
CharacterTable( "3^5:M11" )
gap> s:= CharacterTable( "Suz.2M6" );
CharacterTable( "3^5:(M11x2)" )
gap> CheckConditionsForLemma3( s0, s, facttbl, tbl2, "all" );
2.Suz.2:  3^5:(M11x2) lifts to a direct product,
```

```
proved by squares in [ 1, 4, 8, 10, 19, 22, 26, 39 ].
gap> VerifyCandidates( s, tbl, tbl2, faith[12], "all" );
G = 2.Suz.2:  point stabilizer 3^5:(M11x2), ranks [ 14 ]
[ "1a^++364a^{\\pm}bc+5940a^++12012a^-+14300a^-+15015ab+15795a^++16016ab+54054\
a^++100100a^-b^{\\pm}" ]
gap> faith[8]:= faith[8]{ [ 1, 1 ] };;
gap> faith[12]:= faith[12]{ [ 1, 1 ] };;
gap> CompareWithDatabase( "2.Suz.2", faith );
```

The group $(2.Suz.2)^*$ of the isoclinism type that is not printed in the ATLAS has no faithful multiplicity-free permutation action.

```
gap> tbl2:= IsoclinicTable( tbl, tbl2, facttbl );;
gap> faith:= FaithfulCandidates( tbl2, "Suz.2" );;
gap> CompareWithDatabase( "Isoclinic(2.Suz.2)", faith );
```

3.22 $G = 3.Suz$

The group $3.Suz$ has four faithful multiplicity-free permutation actions, with point stabilizers of the types $G_2(4)$, $U_5(2)$, $2_-^{1+6}.U_4(2)$, and $2^{4+6}:3A_6$, respectively, by Lemma 2.1.

```
gap> tbl:= CharacterTable( "3.Suz" );;
gap> faith:= FaithfulCandidates( tbl, "Suz" );;
1:  subgroup $G_2(4)$, degree 5346 (1 cand.)
4:  subgroup $U_5(2)$, degree 98280 (1 cand.)
5:  subgroup $2^{1+6}_-.U_4(2)$, degree 405405 (1 cand.)
6:  subgroup $2^{4+6}:3A_6$, degree 1216215 (1 cand.)
gap> Maxes( tbl );
[ "3xG2(4)", "3^2.U4(3).2_3'", "3xU5(2)", "3x2^(1+6)_-.U4(2)", "3^6.M11",
  "3xJ2.2", "3x2^(4+6).3A6", "(A4x3.L3(4)).2", "3x2^(2+8):(A5xS3)",
  "3xM12.2", "3.3^(2+4):2(A4x2^2).2", "(3.A6xA5):2", "(3^(1+2):4xA6).2",
  "3xL3(3).2", "3xL3(3).2", "3xL2(25)", "3.A7" ]
gap> VerifyCandidates( CharacterTable( "G2(4)" ), tbl, 0, faith[1], "all" );
G = 3.Suz:  point stabilizer G2(4), ranks [ 7 ]
[ "1a+66ab+780a+1001a+1716ab" ]
gap> VerifyCandidates( CharacterTable( "U5(2)" ), tbl, 0, faith[4], "all" );
G = 3.Suz:  point stabilizer U5(2), ranks [ 14 ]
[ "1a+78ab+143a+364a+1365ab+4290ab+5940a+12012a+14300a+27027ab" ]
gap> VerifyCandidates( CharacterTable( "SuzM4" ), tbl, 0, faith[5], "all" );
G = 3.Suz:  point stabilizer 2^1+6.u4q2, ranks [ 23 ]
[ "1a+66ab+143a+429ab+780a+1716ab+3432a+5940a+6720ab+14300a+18954abc+25025a+42\
900ab+64350cd+66560a" ]
gap> VerifyCandidates( CharacterTable( "SuzM7" ), tbl, 0, faith[6], "all" );
G = 3.Suz:  point stabilizer 2^4+6:3a6, ranks [ 27 ]
[ "1a+364a+780a+1001a+1365ab+4290ab+5940a+12012a+14300a+15795a+25025a+27027ab+\
42900ab+66560a+75075a+85800ab+88452a+100100a+104247ab+139776ab" ]
gap> CompareWithDatabase( "3.Suz", faith );
```

3.23 $G = 3.Suz.2$

The group $3.Suz.2$ has four faithful multiplicity-free permutation actions, with point stabilizers of the types $G_2(4).2$, $U_5(2).2$, $2_-^{1+6}.U_4(2).2$, and $2^{4+6}:3S_6$, respectively. We know from the treatment of $3.Suz$ that we can apply Lemma 2.2.

```
gap> tbl2:= CharacterTable( "3.Suz.2" );;
gap> faith:= FaithfulCandidates( tbl2, "Suz.2" );;
1:  subgroup $G_2(4).2$, degree 5346 (1 cand.)
8:  subgroup $U_5(2).2$, degree 98280 (1 cand.)
10:  subgroup $2^{1+6}_-.U_4(2).2$, degree 405405 (1 cand.)
13:  subgroup $2^{4+6}:3S_6$, degree 1216215 (1 cand.)
gap> Maxes( CharacterTable( "Suz.2" ) );
[ "Suz", "G2(4).2", "3_2.U4(3).(2^2)_{133}", "U5(2).2", "2^(1+6)_-.U4(2).2",
  "3^5:(M11x2)", "J2.2x2", "2^(4+6):3S6", "(A4xL3(4):2_3):2",
  "2^(2+8):(S5xS3)", "M12.2x2", "3^(2+4):2(S4xD8)", "(A6:2_2xA5).2",
  "(3^2:8xA6).2", "L2(25).2_2", "A7.2" ]
gap> VerifyCandidates( CharacterTable( "G2(4).2" ), tbl, tbl2, faith[1], "all" );
G = 3.Suz.2:  point stabilizer G2(4).2, ranks [ 5 ]
[ "1a^++66ab+780a^++1001a^++1716ab" ]
gap> VerifyCandidates( CharacterTable( "U5(2).2" ), tbl, tbl2, faith[8], "all" );
G = 3.Suz.2:  point stabilizer U5(2).2, ranks [ 10 ]
[ "1a^++78ab+143a^-+364a^++1365ab+4290ab+5940a^++12012a^-+14300a^-+27027ab" ]
gap> VerifyCandidates( CharacterTable( "Suz.2M5" ), tbl, tbl2, faith[10], "all" );
G = 3.Suz.2:  point stabilizer 2^(1+6)_-.U4(2).2, ranks [ 16 ]
[ "1a^++66ab+143a^-+429ab+780a^++1716ab+3432a^++5940a^++6720ab+14300a^-+18954a\
^-bc+25025a^++42900ab+64350cd+66560a^+" ]
gap> VerifyCandidates( CharacterTable( "Suz.2M8" ), tbl, tbl2, faith[13], "all" );
G = 3.Suz.2:  point stabilizer 2^(4+6):3S6, ranks [ 20 ]
[ "1a^++364a^++780a^++1001a^++1365ab+4290ab+5940a^++12012a^-+14300a^-+15795a^+\
+25025a^++27027ab+42900ab+66560a^++75075a^++85800ab+88452a^++100100a^++104247a\
b+139776ab" ]
gap> CompareWithDatabase( "3.Suz.2", faith );
```

3.24 $G = 6.Suz$

The group $6.Suz$ has one faithful multiplicity-free permutation action, with point stabilizer of the type $U_5(2)$, by Lemma 2.1.

```
gap> tbl:= CharacterTable( "6.Suz" );;
gap> faith:= FaithfulCandidates( tbl, "2.Suz" );;
1:  subgroup $U_5(2) \rightarrow (Suz,4)$, degree 196560 (1 cand.)
gap> VerifyCandidates( CharacterTable( "U5(2)" ), tbl, 0, faith[1], "all" );
G = 6.Suz:  point stabilizer U5(2), ranks [ 26 ]
[ "1a+12ab+78ab+143a+364abc+924ab+1365ab+4290ab+4368ab+5940a+12012a+14300a+160\
16ab+27027ab+27456ab" ]
gap> CompareWithDatabase( "6.Suz", faith );
```

3.25 $G = 6.Suz.2$

The group $6.Suz.2$ that is printed in the Atlas has two faithful multiplicity-free permutation actions, with point stabilizers of the type $U_5(2).2$ (twice). We know from the treatment of $6.Suz$ that we can apply Lemma 2.2, and get two classes in each case by the treatment of $2.Suz.2$.

```
gap> tbl2:= CharacterTable( "6.Suz.2" );;
gap> faith:= FaithfulCandidates( tbl2, "Suz.2" );;
8:  subgroup $U_5(2).2$, degree 196560 (1 cand.)
gap> VerifyCandidates( CharacterTable( "U5(2).2" ), tbl, tbl2, faith[8], "all" );
G = 6.Suz.2:  point stabilizer U5(2).2, ranks [ 16 ]
```

```
[ "1a^++12ab+78ab+143a^-+364a^+bc+924ab+1365ab+4290ab+4368ab+5940a^++12012a^-+\
14300a^-+16016ab+27027ab+27456ab" ]
gap> faith[8]:= faith[8]{ [ 1, 1 ] };;
gap> CompareWithDatabase( "6.Suz.2", faith );
```

It follows from the treatment of $(2.Suz.2)^*$ that the group $(6.Suz.2)^*$ of the isoclinism type that is not printed in the ATLAS does not have a faithful multiplicity-free permutation action.

```
gap> CompareWithDatabase( "Isoclinic(6.Suz.2)", [] );
```

3.26 $G = 3.ON$

The group $3.ON$ has four faithful multiplicity-free permutation actions, with point stabilizers of the types $L_3(7).2$ (twice) and $L_3(7)$ (twice). (The Schur multiplier of $L_3(7).2$ is trivial, so the $L_3(7)$ type subgroups lift to direct products with the centre of $3.ON$, that is, we can apply Lemma 2.1.)

```
gap> tbl:= CharacterTable( "3.ON" );;
gap> faith:= FaithfulCandidates( tbl, "ON" );;
1:  subgroup $L_3(7).2$, degree 368280 (1 cand.)
2:  subgroup $L_3(7) \leq L_3(7).2$, degree 736560 (1 cand.)
3:  subgroup $L_3(7).2$, degree 368280 (1 cand.)
4:  subgroup $L_3(7) \leq L_3(7).2$, degree 736560 (1 cand.)
gap> VerifyCandidates( CharacterTable( "L3(7).2" ), tbl, 0,
>        Concatenation( faith[1], faith[3] ), "all" );
G = 3.ON:  point stabilizer L3(7).2, ranks [ 11, 11 ]
[ "1a+495ab+10944a+26752a+32395b+52668a+58653bc+63612ab",
  "1a+495cd+10944a+26752a+32395a+52668a+58653bc+63612ab" ]
gap> VerifyCandidates( CharacterTable( "L3(7)" ), tbl, 0,
>        Concatenation( faith[2], faith[4] ), "all" );
G = 3.ON:  point stabilizer L3(7), ranks [ 15, 15 ]
[ "1a+495ab+10944a+26752a+32395b+37696a+52668a+58653bc+63612ab+85064a+122760ab\
",
  "1a+495cd+10944a+26752a+32395a+37696a+52668a+58653bc+63612ab+85064a+122760ab\
" ]
gap> CompareWithDatabase( "3.ON", faith );
```

3.27 $G = 3.ON.2$

The group $3.ON.2$ has no faithful multiplicity-free permutation action.

```
gap> tbl2:= CharacterTable( "3.ON.2" );;
gap> faith:= FaithfulCandidates( tbl2, "ON.2" );;
gap> CompareWithDatabase( "3.ON.2", faith );
```

3.28 $G = 2.Fi_{22}$

The group $2.Fi_{22}$ has seven faithful multiplicity-free permutation actions, with point stabilizers of the types $O_7(3)$ (twice), $O_8^+(2) : S_3$ (twice), $O_8^+(2) : 3$, and $O_8^+(2) : 2$ (twice).

```
gap> tbl:= CharacterTable( "2.Fi22" );;
gap> faith:= FaithfulCandidates( tbl, "Fi22" );;
2:  subgroup $O_7(3)$, degree 28160 (2 cand.)
3:  subgroup $O_7(3)$, degree 28160 (2 cand.)
```

```
4:  subgroup $O_8^+(2):S_3$, degree 123552 (2 cand.)
5:  subgroup $O_8^+(2):3 \leq O_8^+(2):S_3$, degree 247104 (1 cand.)
6:  subgroup $O_8^+(2):2 \leq O_8^+(2):S_3$, degree 370656 (2 cand.)
```

The two classes of maximal subgroups of the type $O_7(3)$ in Fi_{22} induce the same permutation character and lift to two classes of the type $2 \times O_7(3)$ in $2.Fi_{22}$. We get the same two candidates for these two classes. One of them belongs to the first class of $O_7(3)$ subgroups in $2.Fi_{22}$, the other candidate belongs to the second class; this can be seen from the fact that the outer automorphism of Fi_{22} swaps the two classes of $O_7(3)$ subgroups, and the lift of this automorphism to $2.Fi_{22}$ interchanges the candidates –this action can be read off from the embedding of $2.Fi_{22}$ into any group of the type $2.Fi_{22}.2$.

```
gap> faith[2] = faith[3];
true
gap> tbl2:= CharacterTable( "2.Fi22.2" );;
gap> embed:= GetFusionMap( tbl, tbl2 );;
gap> swapped:= Filtered( InverseMap( embed ), IsList );
[ [ 3, 4 ], [ 17, 18 ], [ 25, 26 ], [ 27, 28 ], [ 33, 34 ], [ 36, 37 ],
  [ 42, 43 ], [ 51, 52 ], [ 59, 60 ], [ 63, 65 ], [ 64, 66 ], [ 71, 72 ],
  [ 73, 75 ], [ 74, 76 ], [ 81, 82 ], [ 85, 87 ], [ 86, 88 ], [ 89, 90 ],
  [ 93, 94 ], [ 95, 98 ], [ 96, 97 ], [ 99, 100 ], [ 103, 104 ],
  [ 107, 110 ], [ 108, 109 ], [ 113, 114 ] ]
gap> perm:= Product( List( swapped, pair -> ( pair[1], pair[2] ) ) );;
gap> Permuted( faith[2][1], perm ) = faith[2][2];
true
gap> VerifyCandidates( CharacterTable( "O7(3)" ), tbl, 0, faith[2], "all" );
G = 2.Fi22:  point stabilizer O7(3), ranks [ 5, 5 ]
[ "1a+352a+429a+13650a+13728b", "1a+352a+429a+13650a+13728a" ]
gap> faith[2]:= [ faith[2][1] ];;
gap> faith[3]:= [ faith[3][2] ];;
```

All involutions in Fi_{22} lift to involutions in $2.Fi_{22}$, so the preimages of the maximal subgroups of the type $O_8^+(2).S_3$ in Fi_{22} have the type $2 \times O_8^+(2).S_3$. We apply Lemma 2.3, using that the two subgroups of the type $O_8^+(2).S_3$ contain involutions outside $O_8^+(2)$ which lie in the two nonconjugate preimages of the class 2A of Fi_{22}; this proves the existence of the two candidates of degree 123 552.

```
gap> s:= CharacterTable( "O8+(2).S3" );;
gap> s0:= CharacterTable( "O8+(2).3" );;
gap> facttbl:= CharacterTable( "Fi22" );;
gap> CheckConditionsForLemma3( s0, s, facttbl, tbl, "all" );
2.Fi22:  O8+(2).3.2 lifts to a direct product,
proved by squares in [ 1, 8, 10, 12, 20, 23, 30, 46, 55, 61, 91 ].
gap> derpos:= ClassPositionsOfDerivedSubgroup( s );;
gap> factfus:= GetFusionMap( tbl, facttbl );;
gap> ForAll( PossibleClassFusions( s, tbl ),
>        map -> NecessarilyDifferentPermChars( map, factfus, derpos ) );
true
gap> VerifyCandidates( CharacterTable( "O8+(2).S3" ), tbl, 0, faith[4], "all" );
G = 2.Fi22:  point stabilizer O8+(2).3.2, ranks [ 6, 6 ]
[ "1a+3080a+13650a+13728b+45045a+48048c",
  "1a+3080a+13650a+13728a+45045a+48048b" ]
```

The existence of one class of $O_8^+(2).3$ subgroups follows from Lemma 2.1, and the proof for $O_8^+(2).S_3$ also establishes two classes of $O_8^+(2).2$ subgroups, with different permutation characters,

```
gap> VerifyCandidates( CharacterTable( "O8+(2).3" ), tbl, 0, faith[5], "all" );
G = 2.Fi22:  point stabilizer O8+(2).3, ranks [ 11 ]
[ "1a+1001a+3080a+10725a+13650a+13728ab+45045a+48048bc+50050a" ]
gap> VerifyCandidates( CharacterTable( "O8+(2).2" ), tbl, 0, faith[6], "all" );
G = 2.Fi22:  point stabilizer O8+(2).2, ranks [ 11, 11 ]
[ "1a+352a+429a+3080a+13650a+13728b+45045a+48048ac+75075a+123200a",
  "1a+352a+429a+3080a+13650a+13728a+45045a+48048ab+75075a+123200a" ]
gap> CompareWithDatabase( "2.Fi22", faith );
```

3.29 $G = 2.Fi_{22}.2$

The group $2.Fi_{22}.2$ that is printed in the ATLAS has six faithful multiplicity-free permutation actions, with point stabilizers of the types $O_7(3)$, $O_8^+(2) : S_3$, $O_8^+(2) : 3 \times 2$, $O_8^+(2) : 2$, and ${}^2F_4(2)$ (twice).

```
gap> tbl2:= CharacterTable( "2.Fi22.2" );;
gap> faith:= FaithfulCandidates( tbl2, "Fi22.2" );;
3:  subgroup $O_7(3)$, degree 56320 (1 cand.)
5:  subgroup $O_8^+(2):S_3 \leq O_8^+(2):S_3 \times 2$, degree 247104 (
1 cand.)
6:  subgroup $O_8^+(2):3 \times 2 \leq O_8^+(2):S_3 \times 2$, degree 247104 (
1 cand.)
10:  subgroup $O_8^+(2):2 \leq O_8^+(2):S_3 \times 2$, degree 741312 (1 cand.)
16:  subgroup ${^2F_4(2)^{\prime}}.2$, degree 7185024 (1 cand.)
```

The third, fifth, and tenth multiplicity-free permutation character of $Fi_{22}.2$ are induced from subgroups of the types $O_7(3)$, $O_8^+(2).S_3$, and $O_8^+(2).2$ that lie inside Fi_{22}, and we have discussed above that these groups lift to direct products in $2.Fi_{22}$. In fact all such subgroups of $2.Fi_{22}.2$ lie inside $2.Fi_{22}$, and the two classes of such subgroups in $2.Fi_{22}$ are fused in $2.Fi_{22}.2$, hence we get only one class of these subgroups.

```
gap> VerifyCandidates( CharacterTable( "O7(3)" ), tbl, tbl2, faith[3], "all" );
G = 2.Fi22.2:  point stabilizer O7(3), ranks [ 9 ]
[ "1a^{\\pm}+352a^{\\pm}+429a^{\\pm}+13650a^{\\pm}+13728ab" ]
gap> VerifyCandidates( CharacterTable( "O8+(2).S3" ), tbl, tbl2, faith[5], "all" );
G = 2.Fi22.2:  point stabilizer O8+(2).3.2, ranks [ 10 ]
[ "1a^{\\pm}+3080a^{\\pm}+13650a^{\\pm}+13728ab+45045a^{\\pm}+48048bc" ]
gap> VerifyCandidates( CharacterTable( "O8+(2).2" ), tbl, tbl2, faith[10], "all" );
G = 2.Fi22.2:  point stabilizer O8+(2).2, ranks [ 20 ]
[ "1a^{\\pm}+352a^{\\pm}+429a^{\\pm}+3080a^{\\pm}+13650a^{\\pm}+13728ab+45045a\
^{\\pm}+48048a^{\\pm}bc+75075a^{\\pm}+123200a^{\\pm}" ]
```

The sixth multiplicity-free permutation character of $Fi_{22}.2$ is induced from a subgroup of the type $O_8^+(2).3 \times 2$ that does not lie in Fi_{22}. Let M be a maximal subgroup of $G = 2.Fi_{22}.2$ that maps onto a group of the type $O_8^+(2) : S_3 \times 2$ in the factor group $Fi_{22}.2$. As we have discussed above, any $O_8^+(2).3$ type subgroup of Fi_{22} lifts to a subgroup of the type $2 \times O_8^+(2).3$ in $2.Fi_{22}$, and the outer involutions in the subgroup $O_8^+(2).3 \times 2$ of $Fi_{22}.2$ lift to involutions in $2.Fi_{22}.2$; so M contains two subgroups isomorphic to H that do not contain the centre of $2.Fi_{22}.2$. We use Lemma 2.4 to show that these groups are conjugate in M: The subgroup U has the type $O_8^+(2).3$, the subgroups H and UZ have the type $O_8^+(2) : 3 \times 2$, and so also N/Z has this type.

```
gap> tbl2:= CharacterTable( "2.Fi22.2" );;
gap> facttbl:= CharacterTable( "Fi22.2" );;
gap> tblMbar:= CharacterTable( "O8+(2).S3" ) * CharacterTable( "Cyclic", 2 );;
gap> piMbar:= PossiblePermutationCharacters( tblMbar, facttbl );
```

```
[ Character( CharacterTable( "Fi22.2" ), [ 61776, 6336, 656, 288, 666, 216,
      36, 27, 40, 76, 16, 12, 20, 1, 36, 72, 8, 26, 18, 36, 24, 12, 8, 6, 3,
      1, 4, 8, 0, 2, 6, 3, 0, 1, 1, 0, 4, 10, 4, 4, 0, 0, 4, 2, 4, 3, 0, 1,
      1, 0, 0, 3, 2, 1, 1, 0, 2, 4, 1, 1576, 216, 316, 168, 56, 36, 32, 4,
      46, 64, 10, 16, 10, 30, 10, 1, 9, 6, 4, 4, 8, 0, 6, 1, 1, 1, 24, 6, 6,
      6, 8, 6, 6, 0, 2, 1, 1, 1, 0, 4, 1, 1, 0, 1, 4, 2, 0, 0, 0, 1, 1, 0, 1
     ] ) ]
gap> piM:= piMbar[1]{ GetFusionMap( tbl2, facttbl ) };;
gap> tblNbar:= CharacterTable( "O8+(2).3" ) * CharacterTable( "Cyclic", 2 );;
gap> piNbar:= PossiblePermutationCharacters( tblNbar, facttbl );
[ Character( CharacterTable( "Fi22.2" ), [ 123552, 0, 1312, 192, 1332, 432,
      72, 54, 80, 0, 0, 24, 16, 2, 0, 0, 16, 52, 0, 48, 0, 24, 16, 0, 6, 2,
      4, 4, 0, 0, 12, 6, 0, 0, 2, 0, 8, 20, 8, 0, 0, 0, 0, 4, 0, 6, 0, 0, 2,
      0, 0, 0, 4, 0, 2, 0, 4, 4, 0, 3152, 432, 0, 48, 80, 48, 0, 8, 92, 128,
      20, 0, 20, 60, 0, 2, 18, 12, 0, 4, 4, 0, 0, 2, 0, 2, 24, 12, 12, 0, 8,
      12, 0, 0, 0, 2, 2, 0, 0, 8, 2, 0, 0, 0, 4, 4, 0, 0, 0, 2, 0, 0, 2 ] ) ]
gap> piN:= piNbar[1]{ GetFusionMap( tbl2, facttbl ) };;
gap> tblU:= CharacterTable( "O8+(2).3" );;
gap> piU:= PossiblePermutationCharacters( tblU, tbl2 );
[ Character( CharacterTable( "2.Fi22.2" ), [ 494208, 0, 0, 4608, 640, 384,
      5328, 0, 1728, 0, 288, 0, 216, 0, 160, 0, 0, 96, 0, 32, 8, 0, 0, 0, 0,
      64, 96, 112, 0, 96, 0, 0, 96, 48, 16, 0, 0, 24, 8, 0, 8, 8, 0, 0, 48,
      0, 24, 0, 0, 0, 0, 8, 0, 0, 0, 16, 64, 16, 16, 0, 0, 0, 0, 0, 0, 8, 0,
      24, 0, 0, 0, 0, 8, 0, 0, 0, 0, 0, 0, 16, 0, 8, 0, 0, 0, 8, 8, 0, 0, 0,
      0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
      0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
      0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ] ) ]
gap> ProofOfD8Factor( tbl2, piU[1], piM, piN );
[ [ 91, 101, 104, 110, 114, 116, 124, 130, 135, 138, 146 ], [ 3 ] ]
```

Since also $2.Fi_{22}$ contains subgroups of the type $O_8^+(2).3 \times 2$, we must use `"extending"` as the last argument of `VerifyCandidates`.

```
gap> s:= CharacterTable( "O8+(2).3" ) * CharacterTable( "Cyclic", 2 );;
gap> VerifyCandidates( s, tbl, tbl2, faith[6], "extending" );
G = 2.Fi22.2:  point stabilizer O8+(2).3xC2, ranks [ 9 ]
[ "1a^++1001a^-+3080a^++10725a^++13650a^++13728ab+45045a^++48048bc+50050a^+" ]
```

By Lemma 2.3, the subgroup ${}^2F_4(2)$ of $Fi_{22}.2$ lifts to $2 \times {}^2F_4(2)$ in $2.Fi_{22}.2$; for that, note that the class 4D of ${}^2F_4(2)$ does not lie inside ${}^2F_4(2)'$ and the preimages in $2.Fi_{22}.2$ of the images in $Fi_{22}.2$ square into the subgroup ${}^2F_4(2)'$ of the direct product $2 \times {}^2F_4(2)'$. Since the group $2 \times {}^2F_4(2)$ contains two subgroups of the type ${}^2F_4(2)$, with normalizer $2 \times {}^2F_4(2)$, there are two classes of such subgroups, which induce the same permutation character.

```
gap> facttbl:= CharacterTable( "Fi22.2" );;
gap> s0:= CharacterTable( "2F4(2)'" );;
gap> s:= CharacterTable( "2F4(2)" );;
gap> CheckConditionsForLemma3( s0, s, facttbl, tbl2, "all" );
2.Fi22.2:  2F4(2)'.2 lifts to a direct product,
proved by squares in [ 5, 38, 53 ].
gap> VerifyCandidates( s, tbl, tbl2, faith[16], "all" );
G = 2.Fi22.2:  point stabilizer 2F4(2)'.2, ranks [ 13 ]
[ "1a^++1001a^++1430a^++13650a^++30030a^++133056a^{\\pm}+289575a^-+400400ab+57\
9150a^++675675a^-+1201200a^-+1663200ab" ]
gap> faith[16]:= faith[16]{ [ 1, 1 ] };;
gap> CompareWithDatabase( "2.Fi22.2", faith );
```

The group $(2.Fi_{22}.2)^*$ of the isoclinism type that is not printed in the ATLAS has five faithful multiplicity-free permutation actions, with point stabilizers of the types $O_7(3)$, $O_8^+(2) : S_3$ (twice), and $O_8^+(2) : 2$ (twice).

```
gap> tbl2:= IsoclinicTable( tbl, tbl2, facttbl );;
gap> faith:= FaithfulCandidates( tbl2, "Fi22.2" );;
3:  subgroup $O_7(3)$, degree 56320 (1 cand.)
5:  subgroup $O_8^+(2):S_3 \leq O_8^+(2):S_3 \times 2$, degree 247104 (
1 cand.)
7:  subgroup $O_8^+(2):S_3 \leq O_8^+(2):S_3 \times 2$, degree 247104 (
1 cand.)
10:  subgroup $O_8^+(2):2 \leq O_8^+(2):S_3 \times 2$, degree 741312 (1 cand.)
11:  subgroup $O_8^+(2):2 \leq O_8^+(2):S_3 \times 2$, degree 741312 (1 cand.)
```

The characters arising from the third, fifth, and tenth multiplicity-free permutation character of $Fi_{22}.2$ are induced from subgroups of $2.Fi_{22}$, so these actions have been verified above.

The seventh multiplicity-free permutation character of $Fi_{22}.2$ is induced from a subgroup of the type $O_8^+(2).S_3$ that does not lie in Fi_{22}. By Lemma 2.3 (i), this subgroup lifts to a direct product N in $G = (2.Fi_{22}.2)^*$.

```
gap> tblU:= CharacterTable( "O8+(2).3" );;
gap> tblNbar:= CharacterTable( "O8+(2).S3" );;
gap> CheckConditionsForLemma3( tblU, tblNbar, facttbl, tbl2, "extending" );
Isoclinic(2.Fi22.2):  O8+(2).3.2 lifts to a direct product,
proved by squares in [ 1, 7, 9, 11, 18, 21, 26, 39, 47, 52, 73 ].
```

The G-conjugacy of the two subgroups of the type $O_8^+(2).S_3$ in N follows from Lemma 2.4. Note that there are two permutation characters of G that are induced from $O_8^+(2).S_3$ type subgroups, and the permutation character 1_N^G is determined as the one that does not vanish outside G'.

```
gap> tblNbar:= CharacterTable( "O8+(2).S3" );;
gap> piNbar:= PossiblePermutationCharacters( tblNbar, facttbl );
[ Character( CharacterTable( "Fi22.2" ), [ 123552, 0, 1312, 192, 1332, 432,
      72, 54, 80, 0, 0, 24, 16, 2, 0, 0, 16, 52, 0, 48, 0, 24, 16, 0, 6, 2,
      4, 4, 0, 0, 12, 6, 0, 0, 2, 0, 8, 20, 8, 0, 0, 0, 0, 4, 0, 6, 0, 0, 2,
      0, 0, 0, 4, 0, 2, 0, 4, 4, 0, 0, 0, 632, 288, 32, 24, 64, 0, 0, 0, 0,
      32, 0, 0, 20, 0, 0, 0, 8, 4, 12, 0, 12, 0, 2, 0, 24, 0, 0, 12, 8, 0,
      12, 0, 4, 0, 0, 2, 0, 0, 0, 2, 0, 2, 4, 0, 0, 0, 0, 0, 2, 0, 0 ] ),
  Character( CharacterTable( "Fi22.2" ), [ 123552, 12672, 1312, 576, 1332,
      432, 72, 54, 80, 152, 32, 24, 40, 2, 72, 144, 16, 52, 36, 72, 48, 24,
      16, 12, 6, 2, 8, 16, 0, 4, 12, 6, 0, 2, 2, 0, 8, 20, 8, 8, 0, 0, 8, 4,
      8, 6, 0, 2, 2, 0, 0, 6, 4, 2, 2, 0, 4, 8, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0,
      0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
      0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ] ) ]
gap> piN:= piNbar[1]{ GetFusionMap( tbl2, facttbl ) };;
gap> ProofOfD8Factor( tbl2, piU[1], piM, piN );
[ [ 89, 90, 97, 98, 99, 100, 102, 103, 105, 106, 107, 108, 109, 115, 117,
      119, 127, 128, 129, 132, 133, 134, 145, 149, 150 ], [ 3 ] ]
```

Since also $2.Fi_{22}$ contains subgroups of the type $O_8^+(2) : S_3$, we must use `"extending"` as the last argument of `VerifyCandidates`.

```
gap> s0:= CharacterTable( "O8+(2).3" );;
gap> s:= CharacterTable( "O8+(2).S3" );;
gap> CheckConditionsForLemma3( s0, s, facttbl, tbl2, "extending" );
```

```
Isoclinic(2.Fi22.2):  O8+(2).3.2 lifts to a direct product,
proved by squares in [ 1, 7, 9, 11, 18, 21, 26, 39, 47, 52, 73 ].
gap> VerifyCandidates( s, tbl, tbl2, faith[7], "extending" );
G = Isoclinic(2.Fi22.2):  point stabilizer O8+(2).3.2, ranks [ 9 ]
[ "1a^++1001a^++3080a^++10725a^-+13650a^++13728ab+45045a^++48048bc+50050a^-" ]
```

The existence of exactly one class of $O_8^+(2):2$ type subgroups not contained in $2.Fi_{22}$ follows from the above consideration; the corresponding permutation characters arise from the 11-th multiplicity-free permutation character of $Fi_{22}.2$.

```
gap> s:= CharacterTable( "O8+(2).2" );;
gap> VerifyCandidates( s, tbl, tbl2, faith[11], "extending" );
G = Isoclinic(2.Fi22.2):  point stabilizer O8+(2).2, ranks [ 19 ]
[ "1a^++352a^{\\pm}+429a^{\\pm}+1001a^++3080a^++10725a^-+13650a^++13728ab+4504\
5a^++48048a^{\\pm}bc+50050a^-+75075a^{\\pm}+123200a^{\\pm}" ]
gap> CompareWithDatabase( "Isoclinic(2.Fi22.2)", faith );
```

3.30 $G = 3.Fi_{22}$

The group $3.Fi_{22}$ has six faithful multiplicity-free permutation actions, with point stabilizers of the types $O_8^+(2):S_3$, $O_8^+(2):3$ (twice), $O_8^+(2):2$, $2^6:S_6(2)$, and ${}^2F_4(2)'$.

```
gap> tbl:= CharacterTable( "3.Fi22" );;
gap> faith:= FaithfulCandidates( tbl, "Fi22" );;
4:  subgroup $O_8^+(2):S_3$, degree 185328 (1 cand.)
5:  subgroup $O_8^+(2):3 \leq O_8^+(2):S_3$, degree 370656 (2 cand.)
6:  subgroup $O_8^+(2):2 \leq O_8^+(2):S_3$, degree 555984 (1 cand.)
8:  subgroup $2^6:S_6(2)$, degree 2084940 (1 cand.)
9:  subgroup ${^2F_4(2)^{\prime}}$, degree 10777536 (1 cand.)
```

The preimages of the maximal subgroups of the type $O_8^+(2).S_3$ in Fi_{22} have the type $3 \times O_8^+(2).S_3$, because the Schur multiplier of $O_8^+(2)$ has order 4 and the only central extension of S_3 by a group of order 3 is $3 \times S_3$. Each such preimage contains one subgroup of the type $O_8^+(2).S_3$ with one subgroup of the type $O_8^+(2).3$, two conjugate $O_8^+(2).3$ subgroups which are not contained in $O_8^+(2).S_3$, and one class of $O_8^+(2).2$ subgroups. The two classes of $O_8^+(2).3$ subgroups contain elements of order 3 outside $O_8^+(2)$ which lie in nonconjugate preimages of the class `3A` of Fi_{22}, so we get two classes of $O_8^+(2).3$ subgroups in $3.Fi_{22}$ which induce different permutation characters.

```
gap> VerifyCandidates( CharacterTable( "O8+(2).S3" ), tbl, 0, faith[4], "all" );
G = 3.Fi22:  point stabilizer O8+(2).3.2, ranks [ 10 ]
[ "1a+351ab+3080a+13650a+19305ab+42120ab+45045a" ]
gap> s:= CharacterTable( "O8+(2).3" );;
gap> fus:= PossibleClassFusions( s, tbl );;
gap> facttbl:= CharacterTable( "Fi22" );;
gap> factfus:= GetFusionMap( tbl, facttbl );;
gap> outer:= Difference( [ 1 .. NrConjugacyClasses( s ) ],
>               ClassPositionsOfDerivedSubgroup( s ) );;
gap> outerfus:= List( fus, map -> map{ outer } );
[ [ 13, 13, 18, 18, 46, 46, 50, 50, 59, 59, 75, 75, 95, 95, 98, 98, 95, 95,
      116, 116, 142, 142, 148, 148, 157, 157, 160, 160 ],
  [ 14, 15, 18, 18, 47, 48, 51, 52, 59, 59, 76, 77, 96, 97, 99, 100, 96, 97,
      116, 116, 143, 144, 149, 150, 158, 159, 161, 162 ],
  [ 15, 14, 18, 18, 48, 47, 52, 51, 59, 59, 77, 76, 97, 96, 100, 99, 97, 96,
      116, 116, 144, 143, 150, 149, 159, 158, 162, 161 ] ]
```

```
gap> preim:= InverseMap( factfus )[5];
[ 13, 14, 15 ]
gap> List( outerfus, x -> List( preim, i -> i in x ) );
[ [ true, false, false ], [ false, true, true ], [ false, true, true ] ]
gap> VerifyCandidates( s, tbl, 0, faith[5], "all" );
G = 3.Fi22:  point stabilizer O8+(2).3, ranks [ 11, 17 ]
[ "1a+1001a+3080a+10725a+13650a+27027ab+45045a+50050a+96525ab",
  "1a+351ab+1001a+3080a+7722ab+10725a+13650a+19305ab+42120ab+45045a+50050a+540\
54ab" ]
gap> VerifyCandidates( CharacterTable( "O8+(2).2" ), tbl, 0, faith[6], "all" );
G = 3.Fi22:  point stabilizer O8+(2).2, ranks [ 17 ]
[ "1a+351ab+429a+3080a+13650a+19305ab+27027ab+42120ab+45045a+48048a+75075a+965\
25ab" ]
```

Lemma 2.1 applies to the maximal subgroups of the types $2^6 : S_6(2)$ and ${}^2F_4(2)'$ in Fi_{22} and their preimages in $3.Fi_{22}$.

```
gap> VerifyCandidates( CharacterTable( "2^6:s6f2" ), tbl, 0, faith[8], "all" );
G = 3.Fi22:  point stabilizer 2^6:s6f2, ranks [ 24 ]
[ "1a+351ab+429a+1430a+3080a+13650a+19305ab+27027ab+30030a+42120ab+45045a+7507\
5a+96525ab+123552ab+205920a+320320a+386100ab" ]
gap> VerifyCandidates( CharacterTable( "2F4(2)'" ), tbl, 0, faith[9], "all" );
G = 3.Fi22:  point stabilizer 2F4(2)', ranks [ 25 ]
[ "1a+1001a+1430a+13650a+19305ab+27027ab+30030a+51975ab+289575a+386100ab+40040\
0ab+405405ab+579150a+675675a+1201200a+1351350efgh" ]
gap> CompareWithDatabase( "3.Fi22", faith );
```

3.31 $G = 3.Fi_{22}.2$

The group $3.Fi_{22}.2$ has seven faithful multiplicity-free permutation actions, with point stabilizers of the types $O_8^+(2) : S_3 \times 2$, $O_8^+(2) : 3 \times 2$, $O_8^+(2) : S_3$ (twice), $O_8^+(2) : 2 \times 2$, $2^7 : S_6(2)$, and ${}^2F_4(2)$.

```
gap> tbl2:= CharacterTable( "3.Fi22.2" );;
gap> faith:= FaithfulCandidates( tbl2, "Fi22.2" );;
4:  subgroup $O_8^+(2):S_3 \times 2$, degree 185328 (1 cand.)
6:  subgroup $O_8^+(2):3 \times 2 \leq O_8^+(2):S_3 \times 2$, degree 370656 (
1 cand.)
7:  subgroup $O_8^+(2):S_3 \leq O_8^+(2):S_3 \times 2$, degree 370656 (
2 cand.)
8:  subgroup $O_8^+(2):2 \times 2 \leq O_8^+(2):S_3 \times 2$, degree 555984 (
1 cand.)
9:  subgroup $O_8^+(2):3 \leq O_8^+(2):S_3 \times 2$, degree 741312 (1 cand.)
14:  subgroup $2^7:S_6(2)$, degree 2084940 (1 cand.)
16:  subgroup ${^2F_4(2)^{\prime}}.2$, degree 10777536 (1 cand.)
```

Let H be a subgroup of the type $O_8^+(2) : S_3 \times 2$ in $Fi_{22}.2$; it induces the 4-th multiplicity-free permutation character of $Fi_{22}.2$. The intersection of H with Fi_{22} is of the type $O_8^+(2) : S_3$; it lifts to a direct product in $3.Fi_{22}$, which contains one subgroup of the type $O_8^+(2) : S_3$ that is normal in the preimage of H. By Lemma 2.2, we get one class of subgroups of the type $O_8^+(2) : S_3 \times 2$ in $3.Fi_{22}.2$. The same argument yields one class of each of the types $O_8^+(2) : 3 \times 2$ and $O_8^+(2) : 2 \times 2$, which arise from the 6-th and 8-th multiplicity-free permutation character of $Fi_{22}.2$, respectively.

```
gap> s:= CharacterTable( "O8+(2).S3" ) * CharacterTable( "Cyclic", 2 );;
gap> VerifyCandidates( s, tbl, tbl2, faith[4], "all" );
```

```
G = 3.Fi22.2:  point stabilizer O8+(2).3.2xC2, ranks [ 7 ]
[ "1a^++351ab+3080a^++13650a^++19305ab+42120ab+45045a^+" ]
gap> s:= CharacterTable( "O8+(2).3" ) * CharacterTable( "Cyclic", 2 );;
gap> VerifyCandidates( s, tbl, tbl2, faith[6], "all" );
G = 3.Fi22.2:  point stabilizer O8+(2).3xC2, ranks [ 12 ]
[ "1a^++351ab+1001a^-+3080a^++7722ab+10725a^++13650a^++19305ab+42120ab+45045a^\
++50050a^++54054ab" ]
gap> s:= CharacterTable( "O8+(2).2" ) * CharacterTable( "Cyclic", 2 );;
gap> VerifyCandidates( s, tbl, tbl2, faith[8], "all" );
G = 3.Fi22.2:  point stabilizer O8+(2).2xC2, ranks [ 12 ]
[ "1a^++351ab+429a^++3080a^++13650a^++19305ab+27027ab+42120ab+45045a^++48048a^\
++75075a^++96525ab" ]
```

Let H be a subgroup of the type $O_8^+(2) : S_3$ in $Fi_{22}.2$ that is not contained in Fi_{22}; it induces the 7-th multiplicity-free permutation character of $Fi_{22}.2$. The intersection of H with Fi_{22} is of the type $O_8^+(2) : 3$; it lifts to a direct product in $3.Fi_{22}$, which contains four subgroups of the type $O_8^+(2) : 3$, three of them not containing the centre of $3.Fi_{22}$. By Lemma 2.2, we get three subgroups of the type $O_8^+(2) : S_3$ in $3.Fi_{22}.2$, two of which are conjugate; they induce two different permutation characters, so we get two classes.

(Since there are $O_8^+(2).S_3$ type subgroups also inside $3.Fi_{22}$, we must use `"extending"` as the last argument of `VerifyCandidates`.)

```
gap> s:= CharacterTable( "O8+(2).S3" );;
gap> derpos:= ClassPositionsOfDerivedSubgroup( s );;
gap> facttbl:= CharacterTable( "Fi22.2" );;
gap> sfustbl2:= PossibleClassFusions( s, tbl2,
>        rec( permchar:= faith[7][1] ) );;
gap> ForAll( sfustbl2,
>        map -> NecessarilyDifferentPermChars( map, factfus, derpos ) );
true
gap> VerifyCandidates( s, tbl, tbl2, faith[7], "extending" );
G = 3.Fi22.2:  point stabilizer O8+(2).3.2, ranks [ 9, 12 ]
[ "1a^++1001a^++3080a^++10725a^-+13650a^++27027ab+45045a^++50050a^-+96525ab",
  "1a^++351ab+1001a^++3080a^++7722ab+10725a^-+13650a^++19305ab+42120ab+45045a^\
++50050a^-+54054ab" ]
```

The nineth multiplicity-free permutation character of $Fi_{22}.2$ is induced from a subgroup of the type $O_8^+(2).3$ that lies inside Fi_{22} and is known to lift to s group of the type $3 \times O_8^+(2).3$ in $3.Fi_{22}$. All subgroups of index three in this group either contain the centre of $3.Fi_{22}$ or have the type $O_8^+(2).3$, and it turns out that the permutation characters of $3.Fi_{22}.2$ induced from these subgroups are not multiplicity-free. So the candidate must be excluded.

```
gap> VerifyCandidates( CharacterTable( "O8+(2).3" ), tbl, tbl2, faith[9], "all" );
G = 3.Fi22.2:  no O8+(2).3
gap> faith[9]:= [];;
```

Lemma 2.2 guarantees the existence of one class of subgroups of each of the types $2^7 : S_6(2)$ and ${}^2F_4(2)$.

```
gap> VerifyCandidates( CharacterTable( "2^7:S6(2)" ), tbl, tbl2, faith[14], "all" );
G = 3.Fi22.2:  point stabilizer 2^7:S6(2), ranks [ 17 ]
[ "1a^++351ab+429a^++1430a^++3080a^++13650a^++19305ab+27027ab+30030a^++42120ab\
+45045a^++75075a^++96525ab+123552ab+205920a^++320320a^++386100ab" ]
gap> VerifyCandidates( CharacterTable( "2F4(2)" ), tbl, tbl2, faith[16], "all" );
G = 3.Fi22.2:  point stabilizer 2F4(2)'.2, ranks [ 17 ]
```

```
[ "1a^++1001a^++1430a^++13650a^++19305ab+27027ab+30030a^++51975ab+289575a^-+38\
6100ab+400400ab+405405ab+579150a^++675675a^-+1201200a^-+1351350efgh" ]
gap> CompareWithDatabase( "3.Fi22.2", faith );
```

3.32 $G = 6.Fi_{22}$

The group $6.Fi_{22}$ has six faithful multiplicity-free permutation actions, with point stabilizers of the types $O_8^+(2) : S_3$ (twice), $O_8^+(2) : 3$ (twice), and $O_8^+(2) : 2$ (twice).

```
gap> tbl:= CharacterTable( "6.Fi22" );;
gap> facttbl:= CharacterTable( "3.Fi22" );;
gap> faith:= FaithfulCandidates( tbl, "3.Fi22" );;
1:  subgroup $O_8^+(2):S_3 \rightarrow (Fi_{22},4)$, degree 370656 (2 cand.)
2:  subgroup $O_8^+(2):3 \rightarrow (Fi_{22},5)$, degree 741312 (1 cand.)
3:  subgroup $O_8^+(2):3 \rightarrow (Fi_{22},5)$, degree 741312 (1 cand.)
4:  subgroup $O_8^+(2):2 \rightarrow (Fi_{22},6)$, degree 1111968 (2 cand.)
```

From the discussion of the cases $2.Fi_{22}$ and $3.Fi_{22}$, we conclude that the maximal subgroups of the type $O_8^+(2).S_3$ lift to groups of the type $6 \times O_8^+(2).S_3$ in $6.Fi_{22}$. So Lemma 2.3 (iii) yields two classes of $O_8^+(2) : S_3$ type subgroups, which induce different permutation characters.

```
gap> s:= CharacterTable( "O8+(2).S3" );;
gap> s0:= CharacterTable( "O8+(2).3" );;
gap> CheckConditionsForLemma3( s0, s, facttbl, tbl, "all" );
6.Fi22:  O8+(2).3.2 lifts to a direct product,
proved by squares in [ 1, 22, 28, 30, 46, 55, 76, 104, 131, 141, 215 ].
gap> derpos:= ClassPositionsOfDerivedSubgroup( s );;
gap> factfus:= GetFusionMap( tbl, facttbl );;
gap> ForAll( PossibleClassFusions( s, tbl ),
>        map -> NecessarilyDifferentPermChars( map, factfus, derpos ) );
true
gap> VerifyCandidates( s, tbl, 0, faith[1], "all" );
G = 6.Fi22:  point stabilizer O8+(2).3.2, ranks [ 14, 14 ]
[ "1a+351ab+3080a+13650a+13728b+19305ab+42120ab+45045a+48048c+61776cd",
  "1a+351ab+3080a+13650a+13728a+19305ab+42120ab+45045a+48048b+61776ab" ]
```

Each subgroup of the type $O_8^+(2) : 3$ in $3.Fi_{22}$ lifts to a direct product in $6.Fi_{22}$, which yields one action in each case; since there are two different permutation characters already for $3.Fi_{22}$ (see Section 3.30), we get two different permutation characters induced from $O_8^+(2) : 3$.

```
gap> VerifyCandidates( CharacterTable( "O8+(2).3" ), tbl, 0,
>        Concatenation( faith[2], faith[3] ), "all" );
G = 6.Fi22:  point stabilizer O8+(2).3, ranks [ 17, 25 ]
[ "1a+1001a+3080a+10725a+13650a+13728ab+27027ab+45045a+48048bc+50050a+96525ab+\
123552cd",
  "1a+351ab+1001a+3080a+7722ab+10725a+13650a+13728ab+19305ab+42120ab+45045a+48\
048bc+50050a+54054ab+61776abcd" ]
```

Each subgroup of the type $O_8^+(2) : 2$ in $3.Fi_{22}$ lifts to a direct product in $6.Fi_{22}$, which yields two actions; the permutation characters are different by the argument used for $O_8^+(2) : S_3$.

```
gap> VerifyCandidates( CharacterTable( "O8+(2).2" ), tbl, 0, faith[4], "all" );
G = 6.Fi22:  point stabilizer O8+(2).2, ranks [ 25, 25 ]
[ "1a+351ab+352a+429a+3080a+13650a+13728b+19305ab+27027ab+42120ab+45045a+48048\
ac+61776cd+75075a+96525ab+123200a+123552cd",
```

```
  "1a+351ab+352a+429a+3080a+13650a+13728a+19305ab+27027ab+42120ab+45045a+48048\
ab+61776ab+75075a+96525ab+123200a+123552cd" ]
gap> CompareWithDatabase( "6.Fi22", faith );
```

(Note that the rank 17 permutation character above was missing in the first version of [LM].)

3.33 $G = 6.Fi_{22}.2$

The group $6.Fi_{22}.2$ that is printed in the ATLAS has three faithful multiplicity-free permutation actions, with point stabilizers of the types $O_8^+(2) : 3 \times 2$ and ${}^2F_4(2)$ (twice).

```
gap> tbl2:= CharacterTable( "6.Fi22.2" );;
gap> faith:= FaithfulCandidates( tbl2, "Fi22.2" );;
6:  subgroup $O_8^+(2):3 \times 2 \leq O_8^+(2):S_3 \times 2$, degree 741312 (
1 cand.)
16:  subgroup ${^2F_4(2)^{\prime}}.2$, degree 21555072 (1 cand.)
```

Let M be a maximal subgroup of $6.Fi_{22}.2$ that maps to a subgroup of the type $O_8^+(2).S_3 \times 2$ under the canonical epimorphism to $Fi_{22}.2$. Then the conditions of Lemma 4.1 are satisfied for the factor group F of M modulo the normal subgroup of the type $O_8^+(2)$:

Condition (a) follows from the discussion in Section 3.29. The group $M \cap 6.Fi_{22}$ has the structure $6 \times O_8^+(2).S_3$ (see Section 3.32); this implies that the corresponding index 2 subgroup of F has the structure $6 \times S_3$, which is condition (b). For condition (c), note that the generators of the two direct factors of order 3 in the Sylow 3 subgroup of F are inverted by suitable involutions in F, thus they are commutators and hence the Sylow 3 subgroup lies in F'.

Moreover, we know that M contains subgroups of the type $O_8^+(2).3 \times 2$ that do not lie inside $6.Fi_{22}$ and intersect the centre of $6.Fi_{22}$ trivially, because the factor group $2.Fi_{22}.2$ contains subgroups of this type with the analogous property (see Section 3.29), and the preimages of these groups in $6.Fi_{22}.2$ are split extensions of the normal subgroup of order 3 (see Section 3.32). So we conclude $F \cong G_{72,22}$, and by the above computations, there is exactly one class of $O_8^+(2).3 \times 2$ type subgroups in $6.Fi_{22}.2$ that do not lie in $6.Fi_{22}$.

(Since there are $O_8^+(2).3 \times 2$ type subgroups also inside $6.Fi_{22}$, we must use `"extending"` as the last argument of `VerifyCandidates`.)

```
gap> s:= CharacterTable( "O8+(2).3" ) * CharacterTable( "Cyclic", 2 );;
gap> VerifyCandidates( s, tbl, tbl2, faith[6], "extending" );
G = 6.Fi22.2:  point stabilizer O8+(2).3xC2, ranks [ 16 ]
[ "1a^++351ab+1001a^-+3080a^++7722ab+10725a^++13650a^++13728ab+19305ab+42120ab\
+45045a^++48048bc+50050a^++54054ab+61776abcd" ]
```

The subgroup of the type $6 \times {}^2F_4(2)'$ of $6.Fi_{22}$ extends to $6 \times {}^2F_4(2)$ in $6.Fi_{22}.2$, which contains two subgroups of the type ${}^2F_4(2)$, by Lemma 2.3; so we get two classes of such subgroups, which induce the same permutation character.

```
gap> VerifyCandidates( CharacterTable( "2F4(2)" ), tbl, tbl2, faith[16], "all" );
G = 6.Fi22.2:  point stabilizer 2F4(2)'.2, ranks [ 22 ]
[ "1a^++1001a^++1430a^++13650a^++19305ab+27027ab+30030a^++51975ab+133056a^{\\p\
m}+289575a^-+386100ab+400400ab+405405ab+579150a^++675675a^-+1201200a^-+1351350\
efgh+1663200ab+1796256abcd" ]
gap> faith[16]:= faith[16]{ [ 1, 1 ] };;
gap> CompareWithDatabase( "6.Fi22.2", faith );
```

The group $(6.Fi_{22}.2)^*$ of the isoclinism type that is not printed in the ATLAS has three faithful multiplicity-free permutation actions, with point stabilizers of the type $O_8^+(2) : S_3$ (three times).

```
gap> facttbl:= CharacterTable( "Fi22.2" );;
gap> tbl2:= IsoclinicTable( tbl, tbl2, facttbl );;
gap> faith:= FaithfulCandidates( tbl2, "Fi22.2" );;
7:  subgroup $O_8^+(2):S_3 \leq O_8^+(2):S_3 \times 2$, degree 741312 (
2 cand.)
```

The existence of $O_8^+(2) : S_3$ type subgroups not contained in $6.Fi_{22}$ follows from Lemma 2.2 and the existence of one class of these subgroups in $(2.Fi_{22}.2)^*$; note that we get three complements of the normal subgroup of order 3 in each subgroup of the type $3.O_8^+(2) : S_3$, but Lemma 2.2 does not state anything about the G-conjugacy of these groups.

So we argue as in the case of $6.Fi_{22}.2$, and let M be a maximal subgroup of $(6.Fi_{22}.2)^*$ that maps to a subgroup of the type $O_8^+(2).S_3 \times 2$ under the canonical epimorphism to $Fi_{22}.2$. As above, the conditions of Lemma 4.1 are satisfied for the factor group F of M modulo the normal subgroup of the type $O_8^+(2)$. This time, we conclude $F \cong G_{72,23}$, so there are exactly three classes of $O_8^+(2) : S_3$ type subgroups in $(6.Fi_{22}.2)^*$ that do not lie in $6.Fi_{22}$.

Now the question remains how these three classes of point stabilizers must be mapped to the two possible permutation characters we found above. For that, we first note that by the last statement of Lemma 4.1, the intersections of the point stabilizers with $6.Fi_{22}$ lie in two different conjugacy classes of $O_8^+(2) : 3$ type subgroups of $6.Fi_{22}$. These are the point stabilizers of the two multiplicity-free permutation characters of degree 741 321 that have been established in Section 3.32. This means that the two possible permutation characters are indeed permutation characters.

Which one belongs to *two* multiplicity-free actions of $(6.Fi_{22}.2)^*$? Let us induce the trivial characters of the two relevant point stabilizers in $6.Fi_{22}$ in two steps, first to the maximal subgroup $6 \times O_8^+(2).S_3$ of $6.Fi_{22}$ and then from this group to $6.Fi_{22}$. The two characters obtained in the first step have degree 12, and the one whose extension to $(6.Fi_{22}.2)^*$ belongs to two actions is induced from a *non-normal* $O_8^+(2).3$ type subgroup of $6 \times O_8^+(2).S_3$, whereas the other character is induced from a normal (but noncentral) subgroup of this type.

We execute the first step in the factor group of the type $6 \times S_3$, then inflate the degree 12 characters to $6 \times O_8^+(2).S_3$, and finally induce the these characters to $6.Fi_{22}$.

```
gap> s:= CharacterTable( "O8+(2).S3" ) * CharacterTable( "Cyclic", 6 );;
gap> fact:= s / ClassPositionsOfSolvableResiduum( s );;
gap> Size( fact );
36
gap> OrdersClassRepresentatives( fact );
[ 1, 6, 3, 2, 3, 6, 3, 6, 3, 6, 3, 6, 2, 6, 6, 2, 6, 6 ]
gap> SizesCentralizers( fact );
[ 36, 36, 36, 36, 36, 36, 18, 18, 18, 18, 18, 18, 12, 12, 12, 12, 12, 12 ]
gap> ind:= InducedCyclic( fact, [ 7, 9, 11 ] );;
gap> List( ind, ValuesOfClassFunction );
[ [ 12, 0, 0, 0, 0, 0, 0, 0, 6, 0, 6, 0, 0, 0, 0, 0, 0, 0 ],
  [ 12, 0, 0, 0, 0, 0, 12, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ] ]
```

(The first character has a trivial kernel, so it is the one that is induced from a non-normal subgroup of order three.)

```
gap> rest:= RestrictedClassFunctions( ind, s );;
gap> fus:= PossibleClassFusions( s, tbl );;
gap> Length( fus );
4
gap> ind:= Set( List( fus, map -> Induced( s, tbl, rest, map ) ) );;
gap> Length( ind );
1
gap> rest:= RestrictedClassFunctions( faith[7], tbl );;
```

```
gap> List( ind[1], pi -> Position( rest, pi ) );
[ 1, 2 ]
```

So the induced characters are uniquely determined, and the first of the two characters in `faith[7]` is afforded by two multiplicity-free actions of $(6.Fi_{22}.2)^*$.

```
gap> s:= CharacterTable( "O8+(2).S3" );;
gap> VerifyCandidates( s, tbl, tbl2, faith[7], "extending" );
G = Isoclinic(6.Fi22.2):  point stabilizer O8+(2).3.2, ranks [ 12, 16 ]
[ "1a^++1001a^++3080a^++10725a^-+13650a^++13728ab+27027ab+45045a^++48048bc+500\
50a^-+96525ab+123552cd",
  "1a^++351ab+1001a^++3080a^++7722ab+10725a^-+13650a^++13728ab+19305ab+42120ab\
+45045a^++48048bc+50050a^-+54054ab+61776abcd" ]
gap> faith[7]:= faith[7]{ [ 1, 1, 2 ] };;
gap> CompareWithDatabase( "Isoclinic(6.Fi22.2)", faith );
```

3.34 $G = 2.Co_1$

The group $2.Co_1$ has two faithful multiplicity-free permutation actions, with point stabilizers of the types Co_2 and Co_3, respectively, by Lemma 2.1.

```
gap> tbl:= CharacterTable( "2.Co1" );;
gap> faith:= FaithfulCandidates( tbl, "Co1" );;
1:  subgroup $Co_2$, degree 196560 (1 cand.)
5:  subgroup $Co_3$, degree 16773120 (1 cand.)
gap> VerifyCandidates( CharacterTable( "Co2" ), tbl, 0, faith[1], "all" );
G = 2.Co1:  point stabilizer Co2, ranks [ 7 ]
[ "1a+24a+299a+2576a+17250a+80730a+95680a" ]
gap> VerifyCandidates( CharacterTable( "Co3" ), tbl, 0, faith[5], "all" );
G = 2.Co1:  point stabilizer Co3, ranks [ 12 ]
[ "1a+24a+299a+2576a+17250a+80730a+95680a+376740a+1841840a+2417415a+5494125a+6\
446440a" ]
gap> CompareWithDatabase( "2.Co1", faith );
```

3.35 $G = 3.F_{3+}$

The group $3.F_{3+}$ has two faithful multiplicity-free permutation actions, with point stabilizers of the types Fi_{23} and $O_{10}^-(2)$, respectively, by Lemma 2.1.

```
gap> tbl:= CharacterTable( "3.F3+" );;
gap> faith:= FaithfulCandidates( tbl, "F3+" );;
1:  subgroup $Fi_{23}$, degree 920808 (1 cand.)
2:  subgroup $O_{10}^-(2)$, degree 150532080426 (1 cand.)
gap> VerifyCandidates( CharacterTable( "Fi23" ), tbl, 0, faith[1], "all" );
G = 3.F3+:  point stabilizer Fi23, ranks [ 7 ]
[ "1a+783ab+57477a+249458a+306153ab" ]
gap> VerifyCandidates( CharacterTable( "O10-(2)" ), tbl, 0, faith[2], "all" );
G = 3.F3+:  point stabilizer O10-(2), ranks [ 43 ]
[ "1a+783ab+8671a+57477a+64584ab+249458a+306153ab+555611a+1666833a+6724809ab+1\
9034730ab+35873145a+43779879ab+48893768a+79452373a+195019461ab+203843871ab+415\
098112a+1050717096ab+1264015025a+1540153692a+1818548820ab+2346900864a+32086535\
25a+10169903744a+10726070355ab+13904165275a+15016498497ab+17161712568a+2109675\
1104ab" ]
gap> CompareWithDatabase( "3.F3+", faith );
```

3.36 $G = 3.F_{3+}.2$

The group $3.F_{3+}.2$ has two faithful multiplicity-free permutation actions, with point stabilizers of the types $Fi_{23} \times 2$ and $O_{10}^-(2).2$, respectively, by Lemma 2.2.

```
gap> tbl2:= CharacterTable( "3.F3+.2" );;
gap> faith:= FaithfulCandidates( tbl2, "F3+.2" );;
1:  subgroup $Fi_{23} \times 2$, degree 920808 (1 cand.)
3:  subgroup $O_{10}^-(2).2$, degree 150532080426 (1 cand.)
gap> VerifyCandidates( CharacterTable( "2xFi23" ), tbl, tbl2, faith[1], "all" );
G = 3.F3+.2:  point stabilizer 2xFi23, ranks [ 5 ]
[ "1a^++783ab+57477a^++249458a^++306153ab" ]
gap> VerifyCandidates( CharacterTable( "O10-(2).2" ), tbl, tbl2, faith[3], "all" );
G = 3.F3+.2:  point stabilizer O10-(2).2, ranks [ 30 ]
[ "1a^++783ab+8671a^-+57477a^++64584ab+249458a^++306153ab+555611a^-+1666833a^+\
+6724809ab+19034730ab+35873145a^++43779879ab+48893768a^-+79452373a^++195019461\
ab+203843871ab+415098112a^-+1050717096ab+1264015025a^++1540153692a^++181854882\
0ab+2346900864a^-+3208653525a^++10169903744a^-+10726070355ab+13904165275a^++15\
016498497ab+17161712568a^++21096751104ab" ]
gap> CompareWithDatabase( "3.F3+.2", faith );
```

3.37 $G = 2.B$

The group $2.B$ has one faithful multiplicity-free permutation action, with point stabilizer of the type Fi_{23}, by Lemma 2.1.

```
gap> tbl:= CharacterTable( "2.B" );;
gap> faith:= FaithfulCandidates( tbl, "B" );;
4:  subgroup $Fi_{23}$, degree 2031941058560000 (1 cand.)
gap> VerifyCandidates( CharacterTable( "Fi23" ), tbl, 0, faith[4], "all" );
G = 2.B:  point stabilizer Fi23, ranks [ 34 ]
[ "1a+4371a+96255a+96256a+9458750a+10506240a+63532485a+347643114a+356054375a+4\
10132480a+4221380670a+4275362520a+8844386304a+9287037474a+13508418144a+3665765\
3760a+108348770530a+309720864375a+635966233056a+864538761216a+1095935366250a+4\
322693806080a+6145833622500a+6619124890560a+10177847623680a+12927978301875a+38\
348970335820a+60780833777664a+89626740328125a+110949141022720a+211069033500000\
a+284415522641250b+364635285437500a+828829551513600a" ]
gap> CompareWithDatabase( "2.B", faith );
```

4 Appendix: Explicit Computations with Groups

Only in the proofs for the groups involving M_{22}, explicit computations with the groups were necessary to determine multiplicity-free permutation characters. Additionally, the structure of certain small factor groups of maximal subgroups in extension of Fi_{22} had to be analyzed in order to determine the multiplicity of actions whose existence had been established character-theoretically.

These computations are collected in this appendix.

4.1 $2^4 : A_6$ type subgroups in $2.M_{22}$

We show that the preimage in $2.M_{22}$ of each maximal subgroup of the type $2^4 : A_6$ in M_{22} contains one class of subgroups of the type $2 \times 2^4 : A_5$. For that, we first note that there are two classes of subgroups of the type $2^4 : A_5$ inside $2^4 : A_6$, and that the A_5 subgroups lift to groups of the type $2 \times A_5$ because $2.M_{22}$ does not admit an embedding of $2.A_6$.

```
gap> tbl:= CharacterTable( "2.M22" );;
gap> PossibleClassFusions( CharacterTable( "2.A6" ), tbl );
[  ]
```

Now we fetch a permutation representation of $2.M_{22}$ on 352 points, from the ATLAS of Group Representations (see [WWT+]), via the GAP package AtlasRep (see [WPN+11]), and compute generators for the second class of maximal subgroups, via the straight line program for M_{22}.

```
gap> info:= OneAtlasGeneratingSetInfo( "2.M22", NrMovedPoints, 352 );;
gap> gens:= AtlasGenerators( info.identifier );;
gap> slp:= AtlasStraightLineProgram( "M22", "maxes", 2 );;
gap> sgens:= ResultOfStraightLineProgram( slp.program, gens.generators );;
gap> s:= Group( sgens );;  Size( s );
11520
gap> 2^5 * 360;
11520
```

The subgroup acts intransitively on the 352 points. We switch to the faithful representation on 192 points, and compute the normal subgroup N of order 2^5.

```
gap> orbs:= Orbits( s, MovedPoints( s ) );;
gap> List( orbs, Length );
[ 160, 192 ]
gap> s:= Action( s, orbs[2] );;
gap> Size( s );
11520
gap> syl2:= SylowSubgroup( s, 2 );;
gap> repeat
>   x:= Random( syl2 );
>   n:= NormalClosure( s, SubgroupNC( s, [ x ] ) );
> until Size( n ) = 32;
```

The point stabilizer S in this group has type A_5, and generates together with N one of the desired subgroups of the type $2^5 : A_5$. However, S does not normalize a subgroup of order 2^4, and so there is no subgroup of the type $2^4 : A_5$.

```
gap> stab:= Stabilizer( s, 192 );;
gap> sub:= ClosureGroup( n, stab );;
gap> Size( sub );
1920
gap> Set( List( Elements( n ),
>         x -> Size( NormalClosure( sub, SubgroupNC( sub, [ x ] ) ) ) ) );
[ 1, 2, 32 ]
```

A representative of the other class of A_5 type subgroups can be found by taking an element x of order three that is not conjugate to one in S, and to choose an element y of order five such that the product is an involution.

```
gap> syl3:= SylowSubgroup( s, 3 );;
gap> repeat three:= Random( stab ); until Order( three ) = 3;
gap> repeat other:= Random( syl3 );
>        until Order( other ) = 3 and not IsConjugate( s, three, other );
gap> syl5:= SylowSubgroup( s, 5 );;
gap> repeat y:= Random( syl5 )^Random( s ); until Order( other*y ) = 2;
gap> a5:= Group( other, y );;
```

```
gap> IsConjugate( s, a5, stab );
false
gap> sub:= ClosureGroup( n, a5 );;
gap> Size( sub );
1920
gap> Set( List( Elements( n ),
>          x -> Size( NormalClosure( sub, SubgroupNC( sub, [ x ] ) ) ) ) );
[ 1, 2, 16, 32 ]
```

This proves the existence of one class of the desired subgroups. Finally, we show that the character table of these groups is indeed the one we used in Section 3.3.

```
gap> g:= First( Elements( n ),
>        x -> Size( NormalClosure( sub, SubgroupNC( sub, [ x ] ) ) ) = 16 );;
gap> compl:= ClosureGroup( a5, g );;
gap> Size( compl );
960
gap> tbl:= CharacterTable( compl );;
gap> IsRecord( TransformingPermutationsCharacterTables( tbl,
>         CharacterTable( "P1/G1/L1/V1/ext2" ) ) );
true
```

4.2 $2^4 : S_5$ type subgroups in $M_{22}.2$

A maximal subgroup of the type $2^4 : S_6$ in $M_{22}.2$ is perhaps easiest found as the point stabilizer in the degree 77 permutation representation. In order to find its index 6 subgroups, the degree 22 permutation representation of $M_{22}.2$ is more suitable because the restriction to the $2^4 : S_6$ type subgroup has orbits of the lengths 6 and 16, where the action of the orbit of length 6 is the natural permutation action of S_6.

So we choose the sum of the two representations, of total degree 99. For convenience, we find this representation as the point stabilizer in the degree 100 representation of $HS.2$, which is contained in the Atlas of Group Representations (see [WWT+]).

```
gap> info:= OneAtlasGeneratingSetInfo( "HS.2", NrMovedPoints, 100 );;
gap> gens:= AtlasGenerators( info.identifier );;
gap> stab:= Stabilizer( Group( gens.generators ), 100 );;
gap> orbs:= Orbits( stab, MovedPoints( stab ) );;
gap> List( orbs, Length );
[ 77, 22 ]
gap> pnt:= First( orbs, x -> Length( x ) = 77 )[1];;
gap> m:= Stabilizer( stab, pnt );;
gap> Size( m );
11520
```

Now we find two nonconjugate subgroups of the type $2^4 : S_5$ as the stabilizer of a point and of a total in S_6, respectively (cf. [CCN+85, p. 4]).

```
gap> orbs:= Orbits( m, MovedPoints( m ) );;
gap> List( orbs, Length );
[ 60, 16, 6, 16 ]
gap> six:= First( orbs, x -> Length( x ) = 6 );;
gap> p:= ( six[1], six[2] )( six[3], six[4] )( six[5], six[6] );;
gap> conj:= ( six[2], six[4], six[5], six[6], six[3] );;
gap> total:= List( [ 0 .. 4 ], i -> p^( conj^i ) );;
```

```
gap> stab1:= Stabilizer( m, six[1] );;
gap> stab2:= Stabilizer( m, Set( total ), OnSets );;
gap> IsConjugate( m, stab1, stab2 );
false
```

We identify the character tables of the two groups in the GAP Character Table Library.

```
gap> s1:= CharacterTable( stab1 );;
gap> s2:= CharacterTable( stab2 );;
gap> NrConjugacyClasses( s1 );  NrConjugacyClasses( s2 );
12
18
gap> lib1:= CharacterTable( "2^4:s5" );;
gap> IsRecord( TransformingPermutationsCharacterTables( lib1, s1 ) );
true
gap> lib2:= CharacterTable( "w(d5)" );;
gap> IsRecord( TransformingPermutationsCharacterTables( lib2, s2 ) );
true
```

The first subgroup does not lead to multiplicity-free permutation characters of $2.M_{22}.2$. Note that there are two classes of subgroups of this type in $M_{22}.2$, one of them is contained in M_{22} and the other is not. The action on the cosets of the former is multiplicity-free, but it does not lift to a multiplicity-free candidate of $2.M_{22}.2$; and the action on the cosets of the latter is not multiplicity-free.

```
gap> tbl:= CharacterTable( "M22" );;
gap> tbl2:= CharacterTable( "M22.2" );;
gap> pi:= PossiblePermutationCharacters( s1, tbl2 );
[ Character( CharacterTable( "M22.2" ), [ 462, 30, 12, 2, 2, 2, 0, 0, 0, 0,
      0, 56, 0, 0, 12, 2, 2, 0, 0, 0, 0 ] ),
  Character( CharacterTable( "M22.2" ), [ 462, 46, 12, 6, 6, 2, 4, 0, 0, 2,
      0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ] ) ]
gap> PermCharInfoRelative( tbl, tbl2, pi ).ATLAS;
[ "1a^++21(a^+)^{2}+55a^++154a^++210a^+",
  "1a^{\\pm}+21a^{\\pm}+55a^{\\pm}+154a^{\\pm}" ]
```

So only the second type of $2^4 : S_5$ type subgroups can lift to the multiplicity-free candidate in question, and this situation is dealt with in Section 3.4.

```
gap> pi:= PossiblePermutationCharacters( s2, tbl2 );
[ Character( CharacterTable( "M22.2" ), [ 462, 30, 3, 2, 2, 2, 3, 0, 0, 0, 0,
      28, 20, 4, 8, 1, 2, 0, 1, 0, 0 ] ) ]
gap> PermCharInfoRelative( tbl, tbl2, pi ).ATLAS;
[ "1a^++21a^++55a^++154a^++231a^-" ]
```

4.3 Multiplicities of Multiplicity-Free Actions of $6.Fi_{22}.2$

We collect the information used in Section 3.33 in a lemma.

Lemma 4.1 *Up to isomorphism, there are exactly two groups G of order 72 with the following properties:*

(a) the Sylow 2 subgroup of G is a dihedral group,

(b) G has a normal subgroup isomorphic to $6 \times S_3$, and

(c) G/G' is a 2-group.

In the GAP library of small groups, they have the identifiers `[ 72, 22 ]` *and* `[ 72, 23 ]`*. Let us denote these groups by $G_{72,22}$ and $G_{72,23}$, let G be one of them, and let N be any normal subgroup of G that satisfies condition (b).*

If $G = G_{72,22}$ then there is exactly one conjugacy class of cyclic subgroups of order 6 in G that have trivial intersection with $Z(N)$; if $G = G_{72,23}$ then there are no such subgroups in G.

If $G = G_{72,23}$ then there are exactly three conjugacy classes of nonabelian subgroups of order 6 in G that do not lie in N and have trivial intersection with $Z(N)$; if $G = G_{72,22}$ then there are no such subgroups in G.

Let U_1, U_2, U_3 denote representatives of the three classes of nonabelian subgroups of order 6 in $G_{72,23}$ mentioned above; the Sylow 3 subgroups of these groups are pairwise different, one of them is normal in N and the other two are conjugate in N.

The proof is given by the following calculations using GAP. We use the classification of groups of order 72, which had been obtained in [Neu67]. The groups are available in GAP via the database of small groups, see [BE99].

```
gap> id_d8:= IdGroup( DihedralGroup( 8 ) );;
gap> id_2xs3:= IdGroup( DirectProduct( CyclicGroup(2), SymmetricGroup(3) ) );;
gap> id_6xs3:= IdGroup( DirectProduct( CyclicGroup(6), SymmetricGroup(3) ) );;
gap> grps:= AllSmallGroups( Size, 72,
>               g -> IdGroup( SylowSubgroup( g, 2 ) ) = id_d8 and
>                    ForAny( NormalSubgroups( g ),
>                            n -> IdGroup( n ) = id_6xs3 ) and
>                    ForAll( AbelianInvariants(g), IsEvenInt ), true );
[ <pc group of size 72 with 5 generators>,
  <pc group of size 72 with 5 generators> ]
gap> List( grps, IdGroup );
[ [ 72, 22 ], [ 72, 23 ] ]
gap> is_good_1:= function( R, N )
>    return Size( R ) = 6 and IsCyclic( R ) and
>           Size( Intersection( R, Centre( N ) ) ) = 1;
> end;;
gap> is_good_2:= function( R, N )
>    return Size( R ) = 6 and not IsCyclic( R ) and
>           not IsSubset( N, R ) and
>           Size( Intersection( R, Centre( N ) ) ) = 1;
> end;;
gap> cand:= Filtered( NormalSubgroups( grps[1] ),
>                     n -> IdGroup( n ) = id_6xs3 );;
gap> classreps:= List( ConjugacyClassesSubgroups( grps[1] ),
>                      Representative );;
gap> List( cand, N -> Number( classreps, R -> is_good_1( R, N ) ) );
[ 1, 1 ]
gap> List( cand, N -> Number( classreps, R -> is_good_2( R, N ) ) );
[ 0, 0 ]
gap> cand:= Filtered( NormalSubgroups( grps[2] ),
>                     n -> IdGroup( n ) = id_6xs3 );;
gap> classreps:= List( ConjugacyClassesSubgroups( grps[2] ),
>                      Representative );;
gap> List( cand, N -> Number( classreps, R -> is_good_1( R, N ) ) );
[ 0 ]
gap> List( cand, N -> Number( classreps, R -> is_good_2( R, N ) ) );
```

```
[ 3 ]
gap> N:= cand[1];;
gap> subs:= Filtered( classreps, R -> is_good_2( R, N ) );;
gap> syl3:= List( subs, x -> SylowSubgroup( x, 3 ) );;
gap> Length( Set( syl3 ) );
3
gap> Number( syl3, x -> IsNormal( N, x ) );
1
```

References

[BE99] H. U. Besche and B. Eick, *The groups of order at most 1000 except 512 and 768*, J. Symbolic Comput. **27** (1999), no. 4, 405–413. MR 1681347 (2000c:20002)

[BL96] T. Breuer and K. Lux, *The multiplicity-free permutation characters of the sporadic simple groups and their automorphism groups*, Comm. Algebra **24** (1996), no. 7, 2293–2316. MR 1390375 (97c:20020)

[BM05] T. Breuer and J. Müller, *Character tables of endomorphism rings of multiplicity-free permutation modules of the sporadic simple groups and their cyclic and bicyclic extensions*, http://www.math.rwth-aachen.de/~Juergen.Mueller/mferctbl/mferctbl.html, 2005.

[Bre12] T. Breuer, *The GAP Character Table Library, Version 1.2*, http://www.math.rwth-aachen.de/~Thomas.Breuer/ctbllib, Mar 2012, GAP package.

[CCN+85] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of finite groups*, Oxford University Press, Eynsham, 1985, Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray. MR 827219 (88g:20025)

[GAP11] *GAP – Groups, Algorithms, and Programming, Version 4.5.1*, http://www.gap-system.org, Jul 2011.

[GAP12] *GAP – Groups, Algorithms, and Programming, Version 4.5.0*, http://www.gap-system.org, Apr 2012.

[Höh01] I. Höhler, *Vielfachheitsfreie Permutationsdarstellungen und die Invarianten zugehöriger Graphen*, Examensarbeit, Lehrstuhl D für Mathematik, Rheinisch Westfälische Technische Hochschule, Aachen, Germany, 2001.

[LM] S. A. Linton and S. A. Mpono, *Multiplicity-free permutation characters of covering groups of sporadic simple groups*, preprint.

[Mül03] J. Müller, *On endomorphism rings and character tables*, Habilitationsschrift, Lehrstuhl D für Mathematik, Rheinisch Westfälische Technische Hochschule, Aachen, Germany, 2003.

[Mül08] ———, *On the multiplicity-free actions of the sporadic simple groups*, J. Algebra **320** (2008), no. 2, 910–926. MR 2422321 (2009d:20028)

[Neu67] J. Neubüser, *Die Untergruppenverbände der Gruppen der Ordnungen* ≤ 100 *mit Ausnahme der Ordnungen 64 und 96*, Habilitationsschrift, Universität Kiel, Kiel, Germany, 1967.

[WPN+11] R. A. Wilson, R. A. Parker, S. Nickerson, J. N. Bray, and T. Breuer, *AtlasRep, a GAP Interface to the Atlas of Group Representations, Version 1.5*, http://www.math.rwth-aachen.de/~Thomas.Breuer/atlasrep, Jul 2011, Refereed GAP package.

[WWT+] R. A. Wilson, P. Walsh, J. Tripp, I. Suleiman, R. A. Parker, S. P. Norton, S. Nickerson, S. Linton, J. Bray, and R. Abbott, *ATLAS of Finite Group Representations*, http://brauer.maths.qmul.ac.uk/Atlas/.