Cohomology Products in CRIME

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May 31, 2011

The purpose of this document is to explain the implementation of cohomology products in the CRIME package for GAP. In this document, the composition $g \circ f$ of two functions f and g is the function obtained by applying f first and then g. The symbol $\circlearrowleft$ is used in diagrams to indicate that a polygon either commutes or anticommutes.

Let G be a finite p-group for some prime p and let $k = \mathbb{F}_p$. We write k for the trivial kG-module. We assume that we can calculate a kG-projective resolution P_* of k. In other words, for n as large as we need, we can compute the integers $\{b_m : 0 \le m \le n\}$ the maps $\{\partial_m : 1 \le m \le n\}$ and the map ϵ such that

$$P_n \xrightarrow{\partial_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} k \tag{1}$$

is exact where $P_m = (kG)^{\oplus b_m}$. Later, we will assume also that P_* is minimal, but this assumption isn't needed at this point.

1 Cohomology Products

The following construction is taken from [2]. We begin with two cocycles $f : P_i \to k$ and $g : P_j \to k$. This means that $f \circ \partial_{i+1} = g \circ \partial_{j+1} = 0$. We want to compute the cup product $fg : P_{i+j} \to k$.

We first convert f into an chain map, resulting in the following commutative diagram.

$$
\begin{array}{ccccccccccccc}
P_m & \xrightarrow{\partial_m} & P_{m-1} & \xrightarrow{\partial_{m-1}} & P_{m-2} & \longrightarrow & \cdots & \longrightarrow & P_{i+2} & \xrightarrow{\partial_{i+2}} & P_{i+1} & \xrightarrow{\partial_{i+1}} & P_i \\
\downarrow{f_m} & & \downarrow{f_{m-1}} & & \downarrow{f_{m-2}} & & & & \downarrow{f_{i+2}} & & \downarrow{f_{i+1}} & & \downarrow{f_i} \quad f \\
P_{m-i} & \xrightarrow{\partial_{m-i}} & P_{m-i-1} & \xrightarrow{\partial_{m-i-1}} & P_{m-i-2} & \longrightarrow & \cdots & \longrightarrow & P_2 & \xrightarrow{\partial_2} & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\epsilon} k \longrightarrow 0
\end{array}
\tag{2}
$$

This is done as follows.

1. Define f_i such that $\epsilon \circ f_i = f$. This is possible by projectivity of P_i.

2. Define f_{i+1} such that $\partial_1 \circ f_{i+1} = f_i \circ \partial_{i+1}$. This is possible by projectivity of P_{i+1} since

$$\text{im}\,(f_i \circ \partial_{i+1}) \le \ker\,(\epsilon) = \text{im}\,(\partial_1)$$

as $\epsilon \circ (f_i \circ \partial_{i+1}) = f \circ \partial_{i+1} = 0$ since f is a cocycle.

3. Define f_{i+2} such that $\partial_2 \circ f_{i+2} = f_{i+1} \circ \partial_{i+2}$. This is possible by projectivity of P_{i+2} since

$$\text{im}\,(f_{i+1} \circ \partial_{i+2}) \le \ker\,(\partial_1) = \text{im}\,(\partial_2)$$

as $\partial_1 \circ (f_{i+1} \circ \partial_{i+2}) = f_i \circ \partial_{i+1} \circ \partial_{i+2} = 0$.

4. Define f_m for $m > i + 2$ by recursion such that $\partial_{m-i} \circ f_m = f_{m-1} \circ \partial_m$. This is possible by projectivity of P_m since

$$\text{im}\,(f_{m-1} \circ \partial_m) \le \ker\,(\partial_{m-i-1}) = \text{im}\,(\partial_{m-i})$$

as $\partial_{m-i-1} \circ (f_{m-1} \circ \partial_m) = f_{m-2} \circ \partial_{m-1} \circ \partial_m = 0$.

Then the product fg is calculated as $g \circ f_{i+j}$. The process above is used to compute the multiplication table used by the `CohomologyRing` command and is used to find generators by the `CohomologyGenerators` command.

2   The Yoneda Cocomplex

My understanding of the purpose of the Yoneda Cocomplex is the following. The definition of the Massey product below requires a cocomplex having an associative product. The product defined above, however, is defined only for f and g *cocycles* in $\text{Hom}\,(P_*, k)$. The Yoneda cocomplex Y, on the other hand, has the same cohomology as $\text{Hom}\,(P_*, k)$, but has an associative product defined for all cochains, namely composition. Moreover, we will show that via the isomorphism $\Phi : H^*\,(G, k) \to H^*\,(Y)$, composition in Y agrees with the product defined in Section 1 up to the factor $(-1)^{\deg f \deg g}$. In other words,

$$\Phi\,(fg) = (-1)^{\deg f \deg g}\,\Phi\,(g) \circ \Phi\,(f).$$

The following construction comes from [1].

Definition 1. *For $i \ge 0$ define*

$$Y^i = \prod_{m \ge i} \text{Hom}_{kG}\,(P_m, P_{m-i}).$$

Then an element $f \in Y^i$ is a collection of kG-homomorphisms $\{f_m : P_m \to P_{m-i} : m \ge i\}$ as in the following diagram.

2

$$P_n \xrightarrow{\partial_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_m \xrightarrow{\partial_m} P_{m-1} \xrightarrow{\partial_{m-1}} P_{m-2} \longrightarrow \cdots \longrightarrow P_{i+1} \xrightarrow{\partial_{i+1}} P_i$$

$$\downarrow f_n \qquad \downarrow f_{n-1} \qquad\qquad \downarrow f_m \qquad \downarrow f_{m-1} \qquad \downarrow f_{m-2} \qquad\qquad \downarrow f_{i+1} \qquad \downarrow f_i$$

$$P_{n-i} \xrightarrow{\partial_{n-i}} P_{n-i-1} \longrightarrow \cdots \longrightarrow P_{m-i} \xrightarrow{\partial_{m-i}} P_{m-i-1} \xrightarrow{\partial_{m-i-1}} P_{m-i-2} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\partial_1} P_0$$

$$\tag{3}$$

Diagram (3) is not required to commute.

Definition 2. $Y = \bigoplus_{i \geq 0} Y^i$ *is called the* Yoneda cocomplex *of* P_*. *We write* $\deg(f) = i$ *for* $f \in Y^i$. *Let* $f = \{f_m : m \geq i\} \in Y^i$ *and define*

$$\partial : Y^i \to Y^{i+1}$$

$$f \mapsto \left\{ f_{m-1} \circ \partial_m - (-1)^i \partial_{m-i} \circ f_m : m \geq 1 \right\}.$$

We observe that cocycles in Y are those elements f for which (3) commutes if $\deg f$ is even and anticommutes if $\deg f$ is odd.

Lemma 3. Y *with differentiation* ∂ *is a cocomplex, that is,* $\partial^2 = 0$.

Proof. Let $f \in Y^i$. We will show that $\partial^2 f = 0$ at the point P_m in (3) for $m \geq i + 2 = \deg(\partial^2 f)$. Follow along in the picture.

$$
\begin{aligned}
(\partial(\partial f))_m &= (\partial f)_{m-1} \circ \partial_m - (-1)^{i+1} \partial_{m-i-1} \circ (\partial f)_m \\
&= \left(f_{m-2} \circ \partial_{m-1} - (-1)^i \partial_{m-i-1} \circ f_{m-1} \right) \circ \partial_m \\
&\quad - (-1)^{i+1} \partial_{m-i-1} \circ \left(f_{m-1} \circ \partial_m - (-1)^i \partial_{m-i} \circ f_m \right) \\
&= f_{m-2} \circ \partial_{m-1} \circ \partial_m - \partial_{m-i-1} \circ \partial_{m-i} \circ f_m \\
&= 0
\end{aligned}
$$

∎

Theorem 4. *The cohomology groups of* Y *are* $H^*(G, k)$.

Proof. We will define a group isomorphism $\Phi : H^i(G, k) \to H^i(Y)$.

1. Let $f : P_i \to k$ be a cocycle in $\mathrm{Hom}^i_{kG}(P_*, k)$, so $f \circ \partial_{i+1} = 0$. Define $\Phi(f) = \{f_m : m \geq i\} \in Y^i$ as follows. The element $\Phi(f)$, together with f, is pictured in the following diagram.

$$P_m \xrightarrow{\partial_m} P_{m-1} \xrightarrow{\partial_{m-1}} P_{m-2} \longrightarrow \cdots \longrightarrow P_{i+2} \xrightarrow{\partial_{i+2}} P_{i+1} \xrightarrow{\partial_{i+1}} P_i$$

$$\downarrow f_m \qquad \downarrow f_{m-1} \qquad \downarrow f_{m-2} \qquad\qquad \downarrow f_{i+2} \qquad \downarrow f_{i+1} \qquad \downarrow f_i \searrow^{f}$$

$$P_{m-i} \xrightarrow{\partial_{m-i}} P_{m-i-1} \xrightarrow{\partial_{m-i-1}} P_{m-i-2} \longrightarrow \cdots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} k \longrightarrow 0$$

$$\tag{4}$$

3

(a) Define f_i such that $\epsilon \circ f_i = f$. This is possible by projectivity of P_i.

(b) Define f_{i+1} such that $\partial_1 \circ f_{i+1} = (-1)^i f_i \circ \partial_{i+1}$. This is possible by projectivity of P_{i+1} since

$$\operatorname{im}\left((-1)^i f_i \circ \partial_{i+1}\right) \le \ker(\epsilon) = \operatorname{im}(\partial_1)$$

as $\epsilon \circ \left((-1)^i f_i \circ \partial_{i+1}\right) = (-1)^i f \circ \partial_{i+1} = 0$.

(c) Define f_{i+2} such that $\partial_2 \circ f_{i+2} = (-1)^i f_{i+1} \circ \partial_{i+2}$. This is possible by projectivity of P_{i+2} since

$$\operatorname{im}\left((-1)^i f_{i+1} \circ \partial_{i+2}\right) \le \ker(\partial_1) = \operatorname{im}(\partial_2)$$

as $\partial_1 \circ \left((-1)^i f_{i+1} \circ \partial_{i+2}\right) = f_i \circ \partial_{i+1} \circ \partial_{i+2} = 0$.

(d) Define f_m for $m > i + 2$ by recursion such that $\partial_{m-i} \circ f_m = (-1)^i f_{m-1} \circ \partial_m$. This is possible by projectivity of P_m since

$$\operatorname{im}\left((-1)^i f_{m-1} \circ \partial_m\right) \le \ker(\partial_{m-i-1}) = \operatorname{im}(\partial_{m-i})$$

as $\partial_{m-i-1} \circ \left((-1)^i f_{m-1} \circ \partial_m\right) = f_{m-2} \circ \partial_{m-1} \circ \partial_m = 0$.

This completes the definition of Φ. The maps $\{f_m : m \ge i\}$ defined in Steps 1b-1d above satisfy

$$\partial_{m-i} \circ f_m = (-1)^i f_{m-1} \partial_m.$$

In other words, $(\partial \Phi(f))_{m+1} = 0$ for all $m \ge i + 1$ so that $\partial \Phi(f) = 0$. Thus, $\Phi(f)$ is a cocycle by construction.

2. We claim than any other choice of maps $\{f'_m : m \ge i\}$ satisfying the conditions in 1a-1d above will be equivalent to $\{f_m : m \ge i\}$ in $H^i(Y)$. More precisely, if f and f' both satisfy conditions 1a-1d, then will define a map $\theta \in Y^{i-1}$ such that $\partial \theta = f - f'$. Write $g_m = f_m - f'_m$ for $m \ge i$.

(5)

(e) Take $\theta_{i-1} = 0$.

(f) Since $\epsilon \circ f_i = \epsilon \circ f'_i = f$, we have $\operatorname{im}(g_i) \le \ker(\epsilon) = \operatorname{im}(\partial_1)$. Define θ_i such that $\partial_1 \circ \theta_i = (-1)^i g_i$. This is possible by projectivity of P_i. We rewrite the condition on θ_i for future reference as follows.

$$(\partial \theta)_i = 0 - (-1)^{i-1} \partial_1 \circ \theta_i = g_i \tag{6}$$

4

(g) By (2f), we have

$$\partial_1 \circ \theta_i \circ \partial_{i+1} = (-1)^i g_i \circ \partial_{i+1} = \partial_1 \circ g_{i+1}$$

so that

$$\text{im}\,(g_{i+1} - \theta_i \circ \partial_{i+1}) \leq \ker(\partial_1) = \text{im}\,(\partial_2).$$

Define θ_{i+1} such that

$$\partial_2 \circ \theta_{i+1} = (-1)^i (g_{i+1} - \theta_i \circ \partial_{i+1}),$$

and again, for future reference, we rewrite this as follows.

$$(\partial\theta)_{i+1} = \theta_i \circ \partial_{i+1} - (-1)^{i-1} \partial_2 \circ \theta_{i+1} = g_{i+1} \tag{7}$$

(h) Assume by recursion that we have computed θ_{m-2} and θ_{m-3} such that

$$\partial_{m-i-1} \circ \theta_{m-2} = (-1)^i (g_{m-2} - \theta_{m-3} \circ \partial_{m-2}).$$

Then $\partial_{m-i-1} \circ \theta_{m-2} \circ \partial_{m-1} = (-1)^i g_{m-2} \circ \partial_{m-1} = \partial_{m-i-1} \circ g_{m-1}$ so that

$$\text{im}\,(g_{m-1} - \theta_{m-2} \circ \partial_{m-1}) \leq \ker(\partial_{m-i-1}) = \text{im}\,(\partial_{m-i}).$$

Define θ_{m-1} such that

$$\partial_{m-i} \circ \theta_{m-1} = (-1)^i (g_{m-1} - \theta_{m-2} \circ \partial_{m-1}),$$

and again, for future reference, we rewrite this as follows.

$$(\partial\theta)_{m-1} = \theta_{m-2} \circ \partial_{m-1} - (-1)^{i-1} \partial_{m-i} \circ \theta_{m-1} = g_{m-1} \tag{8}$$

This completes the definition of θ. Then θ satisfies $\partial\theta = f - f'$ by (6), (7), and (8).

3. Suppose now that $f = \partial g$ for some cochain $g : P_{i-1} \to k$. Write $\Phi(g \circ \partial_i) = \{g_m : m \geq i\}$. We will construct θ such that $\Phi(\partial g) = \partial\theta$ for some $\theta \in Y^{i-1}$ as in the following diagram.

(i) Define θ_{i-1} such that $\epsilon \circ \theta_{i-1} = g$. This is possible by projectivity of P_{i-1}.

(j) Since $\epsilon \circ \theta_{i-1} \circ \partial_i = g \circ \partial_i = \epsilon \circ g_i$, we have that

$$\operatorname{im}(g_i - \theta_{i-1} \circ \partial_i) \leq \ker(\epsilon) = \operatorname{im}(\partial_1).$$

Thus, by projectivity of P_i, we have θ_i such that

$$\partial_1 \circ \theta_i = (-1)^i (g_i - \theta_{i-1} \circ \partial_i).$$

Then

$$(\partial\theta)_i = \theta_{i-1} \circ \partial_i - (-1)^{i-1} \partial_1 \circ \theta_i = g_i.$$

(k) Assume by recursion that we have computed the maps θ_{m-1} and θ_{m-2} such that

$$\theta_{m-2} \circ \partial_{m-1} - (-1)^{i-1} \partial_{m-i} \circ \theta_{m-1} = g_{m-1}.$$

Then

$$\partial_{m-i} \circ g_m = (-1)^i g_{m-1} \circ \partial_m = \partial_{m-i} \circ \theta_{m-1} \circ \partial_m$$

so that

$$\operatorname{im}(g_m - \theta_{m-1} \circ \partial_m) \leq \ker(\partial_{m-i}) = \operatorname{im}(\partial_{m-i+1}).$$

Define θ_m such that

$$\partial_{m-i-1} \circ \theta_m = (-1)^i (g_m - \theta_{m-1} \circ \partial_m).$$

Then

$$(\partial\theta)_m = \theta_{m-1} \circ \partial_m - (-1)^{i-1} \partial_{m-i+1} \circ \theta_m = g_m.$$

This completes the definition of θ. Then $g = \partial\theta$ by construction.

4. We will now show that Φ is a k-module homomorphism. Let $f, g : P_i \to k$ be cocycles and let $\alpha, \beta \in k$. Write $h = \alpha f + \beta g$. We want to show that $\Phi(h) = \alpha\Phi(f) + \beta\Phi(g)$. But $\epsilon \circ h_0 = \epsilon \circ (\alpha f_0 + \beta g_0) = \alpha f + \beta g$, so that we are in the situation of Step 2 above. Thus, $\Phi(h)$ and $\alpha\Phi(f) + \beta\Phi(g)$ are equivalent elements of Y.

5. By Steps 3 and 4, we have that if f and f' are equivalent in $H^*(G, k)$, then $\Phi(f)$ and $\Phi(f')$ are equivalent in $H^*(Y)$. This together with 2 shows that Φ is a well-defined k-module homomorphism.

6. Finally, Φ is a bijection, having inverse given by

$$\{f_m : m \geq i\} \mapsto \epsilon \circ f_i.$$

$\blacksquare$

3   Products in Y

In this section we define a product $Y^i \otimes Y^j \to Y^{i+j}$ on Y. If $f \in Y^i$ and $g \in Y^j$ then let fg be element of Y^{i+j} whose component maps are the compositions of the component maps of f with those of g such that legitimate compositions are obtained, as in the following diagram.

$$
\begin{array}{ccccccccccc}
P_n & \xrightarrow{\partial_n} & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_m & \xrightarrow{\partial_m} & P_{m-1} & \longrightarrow \cdots \longrightarrow & P_{i+j+1} & \xrightarrow{\partial_{i+j+1}} & P_{i+j} \\
\downarrow{f_n} & & \downarrow{f_{n-1}} & & & & \downarrow{f_m} & & \downarrow{f_{m-1}} & & \downarrow{f_{i+j+1}} & & \downarrow{f_{i+j}} \\
P_{n-i} & \xrightarrow{\partial_{n-i}} & P_{n-i-1} & \longrightarrow \cdots & & \longrightarrow & P_{m-i} & \xrightarrow{\partial_{m-i}} & P_{m-i-1} & \longrightarrow \cdots \longrightarrow & P_{j+1} & \xrightarrow{\partial_{j+1}} & P_j \\
\downarrow{g_{n-i}} & & \downarrow{g_{n-i-1}} & & & & \downarrow{g_{m-i}} & & \downarrow{g_{m-i-1}} & & \downarrow{g_{j+1}} & & \downarrow{g_j} \\
P_{n-i-j} & \xrightarrow{\partial_{n-i-j}} & P_{n-i-j-1} & \longrightarrow \cdots & & \longrightarrow & P_{m-i-j} & \xrightarrow{\partial_{m-i-j}} & P_{m-i-j-1} & \longrightarrow \cdots \longrightarrow & P_1 & \xrightarrow{\partial_1} & P_0
\end{array}
$$
$$(9)$$

Observe that we have thrown away the maps $\{f_m : i \leq m \leq i + j - 1\}$. I suppose that the natural symbol for the object in (9) would be $g \circ f$, to emphasize the fact that we're talking about the component-wise composition of two elements of Y and *not* a cohomology product.

Observation 5. $\partial (g \circ f) = g \circ \partial f + (-1)^{\deg f} \partial g \circ f$.

Proof. Write $i = \deg(f)$ and $j = \deg(g)$ as in (9). We will show the claim at the point P_m in (9) for $m \geq i + j + 1 = \deg(\partial(g \circ f))$. Follow along in the picture.

$$
\begin{aligned}
\left(g \circ \partial f + (-1)^i \partial g \circ f \right)_m &= g_{m-i-1} \circ \left(f_{m-1} \circ \partial_m - (-1)^i \partial_{m-i} \circ f_m \right) \\
&\quad + (-1)^i \left(g_{m-i-1} \circ \partial_{m-i} - (-1)^j \partial_{m-i-j} \circ g_{m-i} \right) \circ f_m \\
&= g_{m-i-1} \circ f_{m-1} \circ \partial_m - (-1)^{i+j} \partial_{m-i-j} \circ g_{m-i} \circ f_m \\
&= (\partial(g \circ f))_m
\end{aligned}
$$

■

Claim 6. *Composition in Y induces via Φ an associative binary operation*

$$
H^i(G, k) \otimes H^j(G, k) \to H^{i+j}(G, k)
$$

making $H^*(G, k)$ into a ring with 1.

4   Relationships among products on $H^*(G, k)$

Let $f \in H^i(G, k)$ and $g \in H^j(G, k)$. Consider the following products on $H^*(G, k)$.

1. The *cup product* fg defined in Section 1

2. The product induced from composition in Y

$$(f, g) \xrightarrow{\Phi} \big(\Phi(f), \Phi(g) \big) \xrightarrow{\circ} \Phi(g) \circ \Phi(f) \xrightarrow{\Phi^{-1}} \epsilon \circ \big(\Phi(g) \circ \Phi(f) \big)_{i+j}$$

3. The *Massey 2-fold product* $\langle f, g \rangle$, defined more generally in Section 5 below,

$$(f, g) \xrightarrow{\Phi} \big(\Phi(f), \Phi(g) \big) \xrightarrow{\langle \cdot \rangle} (-1)^i \, \Phi(g) \circ \Phi(f) \xrightarrow{\Phi^{-1}} (-1)^i \, \epsilon \circ \big(\Phi(g) \circ \Phi(f) \big)_{i+j}$$

The cup product is calculated as $g \circ f_{i+j}$, where f_{i+j} is as in (2), whereas product 2 is calculated as $g \circ f_{i+j}$, where f_{i+j} is as in (4). Comparing (2) and (4), we see that the two f_i's are the same, the f_{i+1}'s differ by $(-1)^i$, the f_{i+2}'s differ by $(-1)^{2i}$, and in general, the f_{i+m}'s differ by $(-1)^{im}$. Thus, products 1 and 2 differ by $(-1)^{ij}$, that is,

$$\Phi^{-1}\big(\Phi(g) \circ \Phi(f) \big) = (-1)^{ij} \, fg$$

so that

$$\Phi(fg) = (-1)^{ij} \, \Phi(g) \circ \Phi(f)$$

and therefore

$$\Phi(fg) = (-1)^{i(j+1)} \langle f, g \rangle \, .$$

We observe that product 1 is associative (see [2]), and that product 2 is also associative, consisting of composition of functions. The Massey product, however, is not associative in general.

5 Massey Products

The idea of the Massey product is to extend the cohomology product to an n-fold product for $n \geq 2$. The following definition is adapted from [3].

Definition 7. *For $k \geq 2$ let $f^{(1)}, f^{(2)}, \dots, f^{(k)}$ be cocycles in Y. The* Massey k-fold product $\langle f^{(1)}, f^{(2)}, \dots, f^{(k)} \rangle$ *is defined provided that for each pair (i, j) with $1 \leq i < j \leq k$ other than $(1, k)$ the lower-degree product $\langle f^{(i)}, f^{(i+1)}, \dots, f^{(j)} \rangle$ is defined and vanishes as an element of $H^*(Y)$. That is, if for each qualifying (i, j) there exists $u^{i,j} \in Y$ such that $\partial u^{i,j} = \langle f^{(i)}, f^{(i+1)}, \dots, f^{(j)} \rangle$. In this situation, the value of $\langle f^{(1)}, f^{(2)}, \dots, f^{(k)} \rangle$ is defined to be*

$$\sum_{t=1}^{k-1} u^{t+1,k} \circ \overline{u^{1,t}}$$

where the symbols $u^{1,1}$ and $u^{k,k}$ are taken to be $f^{(1)}$ and $f^{(k)}$ respectively and $\overline{u} = (-1)^{\deg(u)} u$.

8

Observe that in the case $k = 2$ the condition on (i, j) is vacuously satisfied, so that $\langle f, g \rangle = g \circ \bar{f}$. Traditionally, one organizes the information in Definition 7 into the array like

$$
\begin{array}{cccc}
f^{(1)} & u^{1,2} & u^{1,3} & \\
& f^{(2)} & u^{2,3} & u^{2,4} \\
& & f^{(3)} & u^{3,4} \\
& & & f^{(4)}
\end{array}
$$

and traces the top row with one hand while tracing the rightmost column with the other hand as t runs from 1 to $k - 1$. In the example shown above we have

$$
\left\langle f^{(1)}, f^{(2)}, f^{(3)}, f^{(4)} \right\rangle = u^{2,4} \circ \overline{f^{(1)}} + u^{3,4} \circ \overline{u^{1,2}} + f^{(4)} \circ \overline{u^{1,3}}.
$$

Lemma 8. $\left\langle f^{(1)}, f^{(2)}, \ldots, f^{(k)} \right\rangle$ *is a cocycle in* Y.

The reason for the sign appearing in Definition 7 becomes apparent in the following proof.

Proof. We begin by making a general observation about Y. Suppose $f \in Y^i$ and that $g = \partial \theta$ for some $\theta \in Y^{j-1}$ as in the following diagram.

Then by Observation 5 we have

$$
\begin{aligned}
(g \circ f)_{i+j+m+1} &= g_{j+m+1} \circ f_{i+j+m+1} \\
&= \theta_{j+m} \circ \partial_{j+m+1} \circ f_{i+j+m+1} - (-1)^{j-1} \partial_{m+2} \circ \theta_{j+m+1} \circ f_{i+j+m+1} \\
&= \theta_{j+m} \circ \partial_{j+m+1} \circ f_{i+j+m+1} - (-1)^{j-1} \partial_{m+2} \circ \theta_{j+m+1} \circ f_{i+j+m+1} \\
&\quad - (-1)^i \theta_{j+m} \circ f_{i+j+m} \circ \partial_{i+j+m+1} + (-1)^i \theta_{j+m} \circ f_{i+j+m} \circ \partial_{i+j+m+1} \\
&= -(-1)^i (\theta \circ (\partial f))_{i+j+m+1} + (-1)^i \partial (\theta \circ f)_{i+j+m+1}
\end{aligned}
$$

so that as elements of $H^*(Y)$ we have

$$
\partial \theta \circ f = -(-1)^i \theta \circ \partial f. \tag{10}
$$

9

Now we compute the derivative of $\left\langle f^{(1)}, f^{(2)}, \ldots, f^{(k)} \right\rangle$.

$$\partial\left(\sum_{t=1}^{k-1} (-1)^{\left(\deg u^{1,t}\right)} u^{t+1,k} \circ u^{1,t}\right) = \sum_{t=1}^{k-1} \left((-1)^{\left(\deg u^{1,t}\right)} u^{t+1,k} \circ \partial u^{1,t} + \partial u^{t+1,k} \circ u^{1,t}\right)$$

$$= \sum_{t=1}^{k-1} \left(-\partial u^{t+1,k} \circ u^{1,t} + \partial u^{t+1,k} \circ u^{1,t}\right)$$

$$= 0$$

∎

Observation 9. *The condition* $\partial u^{i,j} = \left\langle f^{(i)}, f^{(i+1)}, \ldots, f^{(j)} \right\rangle$ *forces*

$$\deg\left(u^{i,j}\right) = \sum_{t=i}^{j} \deg\left(f^{(t)}\right) + i - j$$

and $$\deg\left\langle f^{(i)}, f^{(i+1)}, \ldots, f^{(j)} \right\rangle = \sum_{t=i}^{j} \deg\left(f^{(t)}\right) + i - j + 1.$$

Troubling Observation 10. $\left\langle f^{(1)}, f^{(2)}, \ldots, f^{(k)} \right\rangle$ *is not uniquely defined, unless for each* (i, j) *the condition* $\partial u^{i,j} = \left\langle f^{(i)}, f^{(i+1)}, \ldots, f^{(j)} \right\rangle$ *is satisfied by exactly one cochain* $u^{i,j}$.

Suppose that we are given cocycles $f^{(1)}, f^{(2)}, \ldots, f^{(k)}$ and we want to compute the map $u^{i,j}$ for some (i, j) with $1 \leq i < j \leq k$ other than $(1, k)$. Assume that recursively, we have computed all of the maps in the following array.

$$\begin{array}{cccccc} f^{(i)} & u^{i,i+1} & \cdots & u^{i,j-1} & & \\ & f^{(i+1)} & & u^{i+1,j-1} & u^{i+1,j} & \\ & & & & \vdots & \\ & & & f^{(j-1)} & u^{j-1,j} & \\ & & & & f^{(j)} & \end{array}$$

The map $u^{i,j}$ will be such that

$$\partial u^{i,j} = \left\langle f^{(i)}, f^{(i+1)}, \ldots, f^{(j)} \right\rangle = \sum_{t=i}^{j-1} u^{t+1,j} \circ \overline{u^{i,t}} \tag{11}$$

where $u^{i,i} = f^{(i)}$ and $u^{j,j} = f^{(j)}$. Write g for the map on the right-hand side of (11). Write

$$d = \deg(g) = \sum_{t=i}^{j} \deg\left(f^{(t)}\right) + i - j + 1.$$

The relevant maps are all pictured below.

$$
\begin{array}{ccccccccc}
P_m & \xrightarrow{\ \partial_m\ } & P_{m-1} & \xrightarrow{\ \partial_{m-1}\ } & P_{m-2} & \xrightarrow{\ \partial_{m-2}\ } & P_{m-3} & \longrightarrow & \cdots \\
\ \downarrow{g_m}\ \ {}^{u^{i,j}_{m-1}}\!\!\nearrow & & \downarrow{g_{m-1}}\ \ {}^{u^{i,j}_{m-2}}\!\!\nearrow & & \downarrow{g_{m-2}}\ \ {}^{u^{i,j}_{m-3}}\!\!\nearrow & & \downarrow{g_{m-3}} & & \\
P_{m-d} & \xrightarrow{\ \partial_{m-d}\ } & P_{m-d-1} & \xrightarrow{\ \partial_{m-d-1}\ } & P_{m-d-2} & \xrightarrow{\ \partial_{m-d-2}\ } & P_{m-d-3} & \longrightarrow & \cdots
\end{array}
$$

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & P_{d+2} & \xrightarrow{\ \partial_{d+2}\ } & P_{d+1} & \xrightarrow{\ \partial_{d+1}\ } & P_d & \xrightarrow{\ \partial_d\ } & P_{d-1} \\
 & \downarrow{g_{d+2}}\ \ {}^{u^{i,j}_{d+1}}\!\!\nearrow & & \downarrow{g_{d+1}}\ \ {}^{u^{i,j}_d}\!\!\nearrow & & \circlearrowleft\ \downarrow{g_d}\ \ {}^{u^{i,j}_{d-1}}\!\!\nearrow & & \\
\cdots \longrightarrow & P_2 & \xrightarrow{\ \partial_2\ } & P_1 & \xrightarrow{\ \partial_1\ } & P_0 & \xrightarrow{\ \epsilon\ } & k
\end{array}
$$

We assume now that P_* is *minimal*. In other words, we assume that $\partial_m(P_m) \leq \mathrm{Rad}(P_{m-1})$ for all $m \geq 1$. This implies that $\partial f = 0$ for *any* cochain f, that is, we have $\partial_{i+1} \circ f = 0$ for any kG-homomorphism $f : P_i \to k$.

The map $u^{i,j} \in Y^{d-1}$ is constructed as follows.

1. We take $u^{i,j}_{d-1} = 0$.

2. The assumption that $\langle f^{(i)}, f^{(i+1)}, \ldots, f^{(j)} \rangle = g$ vanishes as an element of $H^d(Y)$ tells us that $\epsilon \circ g_d$ vanishes as an element of $H^d(G, k)$. But since P_* is *minimal*, this means that $\epsilon \circ g_d$ is actually the zero map. Then by projectivity of P_d, there exists $u^{i,j}_d$ such that $\partial_1 \circ u^{i,j}_d = (-1)^d g_d$. Observe that this means

$$\left(\partial u^{i,j}\right)_d = 0 - (-1)^{d-1} \partial_1 \circ u^{i,j}_d = g_d.$$

3. The map g is a cocycle by Lemma 8. This means that the *rectangles* in (**??**) either commute or anticommute, depending on whether d is even or odd. Thus,

$$\partial_1 \circ \left(g_{d+1} - u^{i,j}_d \circ \partial_{d+1}\right) = \partial_1 \circ g_{d+1} - (-1)^d g_d \circ \partial_{d+1} = 0$$

so that

$$\mathrm{im}\left(g_{d+1} - u^{i,j}_d \circ \partial_{d+1}\right) \leq \ker(\partial_1) = \mathrm{im}(\partial_2).$$

Thus, there exists $u^{i,j}_{d+1}$ such that

$$\partial_2 \circ u^{i,j}_{d+1} = (-1)^d \left(g_{d+1} - u^{i,j}_d \circ \partial_{d+1}\right).$$

Observe that this means

$$\left(\partial u^{i,j}\right)_{d+1} = u^{i,j}_d \circ \partial_{d+1} - (-1)^{d-1} \partial_2 \circ u^{i,j}_{d+1} = g_{d+1}.$$

11

4. Assume by recursion that we have constructed that maps $u^{i,j}_{m-2}$ and $u^{i,j}_{m-3}$ such that

$$\partial_{m-d-1} \circ u^{i,j}_{m-2} = (-1)^d \left(g_{m-2} - u^{i,j}_{m-3} \circ \partial_{m-2} \right).$$

Thus

$$\partial_{m-d-1} \circ \left(g_{m-1} - u^{i,j}_{m-2} \circ \partial_{m-1} \right) = \partial_{m-d-1} \circ g_{m-1} - (-1)^d g_{m-2} \circ \partial_{m-1} = 0$$

so that

$$\mathrm{im} \left(g_{m-1} - u^{i,j}_{m-2} \circ \partial_{m-1} \right) \le \ker \left(\partial_{m-d-1} \right) = \mathrm{im} \left(\partial_{m-d} \right).$$

Thus, there exists $u^{i,j}_{m-1}$ such that

$$\partial_{m-d} \circ u^{i,j}_{m-1} = (-1)^d \left(g_{m-1} - u^{i,j}_{m-2} \circ \partial_{m-1} \right).$$

Observe that this means

$$\left(\partial u^{i,j} \right)_{m-1} = u^{i,j}_{m-2} \circ \partial_{m-1} - (-1)^{d-1} \partial_{m-d} \circ u^{i,j}_{m-1} = g_{m-1}.$$

This completes the construction of $u^{i,j}$. By construction, we have $\partial \left(u^{i,j} \right) = g$.

Finally, observe that in the last step in the calculation of $\langle f^{(1)}, f^{(2)}, \ldots, f^{(k)} \rangle$, which is actually the *first* step, as this is a recursive process, it is only necessary to calculate $u^{1,k-1}$, but none of the maps $u^{1,m}$ for $2 \le m \le k-2$, and none of the maps $u^{m,k}$ for $2 \le m \le k-1$. In effect, the sum

$$\sum_{t=1}^{k-1} u^{t+1,k} \circ \overline{u^{1,t}} = \sum_{t=1}^{k-2} u^{t+1,k} \circ \overline{u^{1,t}} + f^{(k)} \circ \overline{u^{1,k-1}}$$

appearing in Definition 7 is calculated as

$$\sum_{t=1}^{k-2} \boxed{u^{t+1,k}_{\deg u^{t+1,k}}} \circ \overline{u^{1,t}_{\deg u^{t+1,k} + \deg u^{1,t}}} + f^{(k)}_{\deg f^{(k)}} \circ \overline{u^{1,k-1}_{\deg f^{(k)} + \deg u^{1,k-1}}},$$

But $u^{t+1,k}_{\deg u^{t+1,k}} = 0$ by construction (see Step 1 above), so the sum reduces to a single term. This is not the case with the intermediate maps $u^{i,j}$ with $j - i \le k - 2$.

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