The SglPPow package

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1 Introduction

SglPPow is a GAP package which extends the GAP small groups library. Currently the small groups library gives access to the following groups:

- Those of order at most 2000 except 1024 (423,164,062 groups);
- Those of cubefree order at most 50,000 (395,703 groups);
- Those of order p^7 for the primes $p = 3, 5, 7, 11$ (907,489 groups);
- Those of order p^n for $n \leq 6$ and all primes p;
- Those of order pq^n where q^n divides 28, 36, 55 or 74 and p is an arbitrary prime not equal to q;
- Those of squarefree order;
- Those whose order factorizes into at most 3 primes.

This package gives access to the groups of order p^7 for $p > 11$, and to the groups of order 3^8.

To access the groups of order p^7 for $p > 11$ you need to attach Bettina Eick and Michael Vaughan-Lee's GAP package LiePRing, and Willem de Graaf's GAP package LieRing. So to fully utilize SglPPow you need to enter the following GAP commands: RequirePackage("sglppow"), RequirePackage("LiePRing"), RequirePackage("LieRing").

The groups of order 3^8 are then available via Michael Vaughan-Lee's database of the groups of order 3^8, and the groups of order p^7 for $p > 11$ are available via Bettina Eick and Michael Vaughan-Lee's database of the nilpotent Lie rings of order p^k ($k \leq 7$, $p > 3$). The groups are obtained from the Lie rings using Willem de Graaf's implementation of the Baker-Campbell-Hausdorff formula.

2 Accessing the data

When this package is loaded, then the groups of order 3^8 and p^7 for $p > 11$ are additionally available via the SmallGroups library. As a result, all groups of order p^n with $p = 2$ and $n \leq 9$ and $p = 3$ and $n \leq 8$ and p arbitrary and $n \leq 7$ are then available via the small groups library.

SmallGroup(size, number)

NumberSmallGroups(size)
SmallGroupsInformation(size)
yield the corresponding information.

There is no IdGroup function available for this extension of the small groups library.

However the user should be aware that there are there are 1,396,077 groups of order 3^8, 1,600,573 groups of order 13^7, and 5,546,909 groups of order 17^7. For general p the number of groups of order p^7 is of order $3p^5$. Furthermore as p increases, the time taken to generate a complete list of the groups of order p^7 grows rapidly. Experimentally the time seems to be proportional to $p^{6.2}$. For $p = 17$, for example, the time taken to generate the complete list is ?????????????. In view of the sizes of these databases, it may be helpful for the user to have some understanding of how they are organized, and so in the following sections we attempt to give a brief overview of the databases.

3 The p-group generation algorithm

The p-group generation algorithm was developed and implemented by Eamonn O'Brien, and we refer the reader to [1] for a detailed description of the algorithm. Briefly, let P be a p-group. The algorithm uses the lower p-central series, defined recursively by $\mathcal{P}_1(P) = P$ and $\mathcal{P}_{i+1}(P) = [\mathcal{P}_i(P), P]\mathcal{P}_i(P)^p$ for $i \geq 1$. The p-class of P is the length of this series. Each p-group P, apart from the elementary abelian ones, is an *immediate descendant* of the quotient P/R where R is the last non-trivial term of the lower p-central series of P. Thus all the groups with order 3^8, except the elementary abelian one, are immediate descendants of groups with order 3^k for $k < 8$. All of the immediate descendants of a p-group Q are quotients of a certain extension of Q; the isomorphism problem for these descendants is equivalent to the problem of determining orbits of certain subgroups of this extension under an action of the automorphism group of Q. Not all p-groups have immediate descendants, those that do are called *capable*, and those which do not are called *terminal*. O'Brien and Vaughan-Lee's classification of the groups of order p^7 [2] is based on a classification of the nilpotent Lie rings of order p^7, and the groups of order p^7 are obtained from the Lie rings using the Baker-Campbell-Hausdorff formula. O'Brien and Vaughan-Lee classified the nilpotent Lie rings of order p^7 using the nilpotent Lie ring generation algorithm, which is a direct analogue of the p-group generation algorithm. Thus the databases of nilpotent Lie rings of order p^7 and of the groups of order 3^8 are organized according to these algorithms: the immediate descendants of order p^7 of each nilpotent Lie ring of order less that p^7 are grouped together in the database of nilpotent Lie rings, and the immediate descendants of order 3^8 of each group of order less than 3^8 are grouped together in the database of groups of order 3^8.

4 The groups of order 3^8

The database of groups of order 3^8 is organized according to rank and p-class. Here rank is the rank of the Frattini quotient, i.e. the size of a minimal generating set, and p-class is as defined in the previous section. The following table gives the number of groups of order 3^8 of each rank and p-class, with the (i, j) entry corresponding to rank i and p-class j.

0	0	0	0	0	0	0	1
0	0	58	486	1343	330	9	0
0	4	216747	40521	2163	24	0	0
0	23361	494666	22343	51	0	0	0
0	578478	14796	80	0	0	0	0
0	566	39	0	0	0	0	0
0	10	0	0	0	0	0	0
1	0	0	0	0	0	0	0

The following table gives the range of numbers used to access the groups of order 3^8 of a given rank and class.

							1
		2–59	60–545	546–1888	1889–2218	2219-2227	
	2228–2231	2232–218978	218979–259499	259500–261662	261663–261686		
	261687–285047	285048–779713	779714–802056	802057–802107			
	802108–1380585	1380586–1395381	1395382–1395461				
	1395462–1396027	1396028–1396066					
	1396067–1396076						
1396077							

As mentioned above, the database is organized according to the p-group generation algorithm. For example, the 9 groups of rank 2, p-class 7, and order 3^8 are numbered from 2219–2227. The groups numbered 2219 and 2220 are descendants of SmallGroup$(3^7, 384)$, and the groups numbered 2221–2227 are descendants of SmallGroup$(3^7, 386)$. Similarly, the 24 groups of rank 3, p-class 6, and order 3^8 are numbered from 261663–261686. The first four of these groups are descendants of SmallGroup$(3^7, 5841)$, the next 17 are descendants of SmallGroup$(3^7, 5844)$, and the last 4 are descendants of SmallGroup$(3^7, 5849)$.

5 The groups of order p^7 $(p > 11)$

The groups of order p^7 for $p > 11$ are obtained from the LiePRing database of nilpotent Lie rings of order p^7 using Willem de Graaf's implementation of the Baker-Campbell-Hausdorff formula. The LiePRing database is organized according to the output from the nilpotent Lie ring generation algorithm. For any given p the first Lie ring in the database is the cyclic Lie ring of order p^7. Next come the two generator Lie rings, then the three generator Lie rings, and so on, ending with the six generator Lie rings, and then finally the elementary abelian Lie ring of rank 7. The first four of the two generator nilpotent Lie rings of order p^7 are immediate descendants of the Lie ring

$$\langle a, b \,|\, pb, \text{ class } 3\rangle$$

of order p^4. The next $p^2 + 8p + 25$ are immediate descendants of the Lie ring

$$\langle a, b \,|\, baa, bab, pba, \text{ class } 3\rangle$$

of order p^5, and the next $p + 6 + (p^2 + 3p + 10)\gcd(p - 1, 3)$ are immediate descendants of

$$\langle a, b \,|\, babb, pa, pb, \text{ class } 4\rangle.$$

And so on. The nine rank 6 Lie rings in the database are the rank 6, p-class 2 Lie rings. These are the immediate descendants of the elementary abelian Lie ring of rank 6.

There is a complete list of presentations for the nilpotent Lie rings of order p^k for $k \leq 7$ valid for all $p > 3$ in the document p567.pdf supplied with the documentation for the LiePRing package. The presentations are grouped as described above, with each group of presentations giving the immediate descendants of a Lie ring of smaller order.

In a few cases the descendants of a parametrized family of Lie rings are grouped together. For example there is a family of $p(p-1)$ distinct Lie rings with presentations of the form

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - \lambda baa - \mu bab,\ pb + \nu baa + \xi bab,\ pc,\ \text{class } 3\rangle$$

with $\lambda, \mu, \nu, \xi \neq 0$. Most of these algebras are terminal, but $\frac{5}{2}p - \frac{9}{2} + \frac{1}{2}\gcd(p-1, 4)$ of them are capable, and they have a total of $\frac{1}{2}p^3 + 2p^2 - 5p + \frac{1}{2} + \frac{p}{2}\gcd(p-1, 4)$ descendants of order p^7 and p-class 4. These descendants have presentations

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa - \mu bab - ybaaa,\ pb + \nu baa + bab - zbaaa,\ pc - tbaaa,\ \text{class } 4\rangle,$$

$$\langle a, b, c \,|\, ca - baa,\ cb,\ pa - baa - \mu bab - ybaaa,\ pb + \nu baa + \mu\nu bab - zbaaa,\ pc - tbaaa,\ \text{class } 4\rangle$$

for various choices of the parameters μ, ν, y, z, t. For any given value of p the $\frac{1}{2}p^3 + 2p^2 - 5p + \frac{1}{2} + \frac{p}{2}\gcd(p-1, 4)$ distinct Lie algebras with presentations of this form are grouped together, with consecutive numbering.

There is no easy way to determine the numbering of (say) the three generator Lie rings of p-class 4, since the numbers depend on p in a very complicated way. A user who wants to access a particular group of descendants as described in p567.pdf is advised to use the LiePRing package directly, as this package also has an option to obtain the corresponding groups via Willem de Graaf's implementation of the Baker-Campbell-Hausdorff formula.

References

[1] E.A. O'Brien, *The p-group generation algorithm*, J.Symbolic Computation **9** (1990), 677–698.

[2] E.A. O'Brien and M.R. Vaughan-Lee, *The groups with order p^7 for odd prime p*, J. Algebra **292** (2005), 243–358.