GradedModules

A **homalg** based package for the Abelian category of finitely presented graded modules over a computable graded ring

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(*this manual is still under construction*)

This manual is best viewed as an HTML document. The latest version is available ONLINE at:

http://homalg.math.rwth-aachen.de/~markus/GradedModules/chap0.html

An OFFLINE version should be included in the documentation subfolder of the package. This package is part of the **homalg**-project:

http://homalg.math.rwth-aachen.de/index.php/unreleased/gradedmodules

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[hpa10]

Chapter 2

Installation of the **GradedModules** Package

To install this package just extract the package's archive file to the GAP pkg directory.

By default the GradedModules package is not automatically loaded by GAP when it is installed. You must load the package with

```
LoadPackage("GradedModules");
```

before its functions become available.

Please, send me an e-mail if you have any questions, remarks, suggestions, etc. concerning this package. Also, I would be pleased to hear about applications of this package.

Mohamed Barakat

Chapter 3

Quick Start

Chapter 4

Ring Maps

4.1 Ring Maps: Attributes

4.1.1 KernelSubobject

▷ KernelSubobject(*phi*) (method)
    Returns: a homalg submodule
    The kernel ideal of the ring map *phi*.

4.2 Ring Maps: Operations and Functions

4.2.1 SegreMap

▷ SegreMap(*R, s*) (method)
    Returns: a homalg ring map
    The ring map corresponding to the Segre embedding of $MultiProj(R)$ into the projective space according to $P(W_1) \times P(W_2) \to P(W_1 \otimes W_2)$.

4.2.2 PlueckerMap

▷ PlueckerMap(*l, n, A, s*) (method)
    Returns: a homalg ring map
    The ring map corresponding to the Plücker embedding of the Grassmannian $G_l(P^n(A)) = G_l(P(W))$ into the projective space $P(\bigwedge^l W)$, where $W = V^*$ is the A-dual of the free module $V = A^{n+1}$ of rank $n+1$.

4.2.3 VeroneseMap

▷ VeroneseMap(*n, d, A, s*) (method)
    Returns: a homalg ring map
    The ring map corresponding to the Veronese embedding of the projective space $P^n(A) = P(W)$ into the projective space $P(S^d W)$, where $W = V^*$ is the A-dual of the free module $V = A^{n+1}$ of rank $n+1$.

Chapter 5

GradedModules

5.1 GradedModules: Category and Representations

5.2 GradedModules: Constructors

5.3 GradedModules: Properties

For more properties see the corresponding section (**Modules: Modules: Properties**)) in the documentation of the homalg package.

5.4 GradedModules: Attributes

5.4.1 BettiTable (for modules)

▷ BettiTable(*M*)   (attribute)

  Returns: a homalg diagram

  The Betti diagram of the homalg graded module *M*.

5.4.2 CastelnuovoMumfordRegularity

▷ CastelnuovoMumfordRegularity(*M*)   (attribute)

  Returns: an integer

  The Castelnuovo-Mumford regularity of the homalg graded module *M*.

5.4.3 CastelnuovoMumfordRegularityOfSheafification

▷ CastelnuovoMumfordRegularityOfSheafification(*M*)   (attribute)

  Returns: an integer

  The Castelnuovo-Mumford regularity of the sheafification of homalg graded module *M*.

  For more attributes see the corresponding section (**Modules: Modules: Attributes**)) in the documentation of the homalg package.

5.5 LISHV: Logical Implications for GradedModules

5.6 GradedModules: Operations and Functions

5.6.1 MonomialMap

▷ MonomialMap(*d, M*)   (operation)

Returns: a homalg map

The map from a free graded module onto all degree *d* monomial generators of the finitely generated homalg module *M*.

```
 _____ Example _____
  gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
  gap> S := GradedRing( R );;
  gap> M := HomalgMatrix( "[ x^3, y^2, z,   z, 0, 0 ]", 2, 3, S );;
  gap> M := LeftPresentationWithDegrees( M, [ -1, 0, 1 ] );
  <A graded non-torsion left module presented by 2 relations for 3 generators>
  gap> m := MonomialMap( 1, M );
  <A homomorphism of graded left modules>
  gap> Display( m );
  x^2,0,0,
  x*y,0,0,
  x*z,0,0,
  y^2,0,0,
  y*z,0,0,
  z^2,0,0,
  0,  x,0,
  0,  y,0,
  0,  z,0,
  0,  0,1

  the graded map is currently represented by the above 10 x 3 matrix

  (degrees of generators of target: [ -1, 0, 1 ])
```

5.6.2 RandomMatrix

▷ RandomMatrix(*S, T*)   (operation)

Returns: a homalg matrix

A random matrix between the graded source module *S* and the graded target module *T*.

```
 _____ Example _____
  gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "a,b,c";;
  gap> S := GradedRing( R );;
  gap> rand := RandomMatrix( S^1 + S^2, S^2 + S^3 + S^4 );
  <A 2 x 3 matrix over a graded ring>
  gap> #Display( rand );
  gap> #-3*a-b,                                                    -1,
  gap> #-a^2+a*b+2*b^2-2*a*c+2*b*c+c^2,                            -a+c,
  gap> #-2*a^3+5*a^2*b-3*b^3+3*a*b*c+3*b^2*c+2*a*c^2+2*b*c^2+c^3,-3*b^2-2*a*c-2*b*c+c^2
```

5.6.3 GeneratorsOfHomogeneousPart

▷ GeneratorsOfHomogeneousPart(*d, M*) (operation)

Returns: a homalg matrix

The resulting homalg matrix consists of a generating set (over *R*) of the *d*-th homogeneous part of the finitely generated homalg *S*-module *M*, where *R* is the coefficients ring of the graded ring *S* with $S_0 = R$.

```
                         ———————— Example ————————
  gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
  gap> S := GradedRing( R );;
  gap> M := HomalgMatrix( "[ x^3, y^2, z,   z, 0, 0 ]", 2, 3, S );;
  gap> M := LeftPresentationWithDegrees( M, [ -1, 0, 1 ] );
  <A graded non-torsion left module presented by 2 relations for 3 generators>
  gap> m := GeneratorsOfHomogeneousPart( 1, M );
  <An unevaluated non-zero 7 x 3 matrix over a graded ring>
  gap> Display( m );
  x^2,0,0,
  x*y,0,0,
  y^2,0,0,
  0,  x,0,
  0,  y,0,
  0,  z,0,
  0,  0,1
  (over a graded ring)
```

Compare with MonomialMap (5.6.1).

5.6.4 SubmoduleGeneratedByHomogeneousPart

▷ SubmoduleGeneratedByHomogeneousPart(*d, M*) (operation)

Returns: a homalg module

The submodule of the homalg module *M* generated by the image of the *d*-th monomial map ($\rightarrow$ MonomialMap (5.6.1)), or equivalently, by the generating set of the *d*-th homogeneous part of *M*.

```
                         ———————— Example ————————
  gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
  gap> S := GradedRing( R );;
  gap> M := HomalgMatrix( "[ x^3, y^2, z,   z, 0, 0 ]", 2, 3, S );;
  gap> M := LeftPresentationWithDegrees( M, [ -1, 0, 1 ] );
  <A graded non-torsion left module presented by 2 relations for 3 generators>
  gap> n := SubmoduleGeneratedByHomogeneousPart( 1, M );
  <A graded left submodule given by 7 generators>
  gap> Display( M );
  z, 0,    0,
  0, y^2*z,z^2,
  x^3,y^2,  z

  Cokernel of the map

  Q[x,y,z]^(1x3) --> Q[x,y,z]^(1x3),

  currently represented by the above matrix
  (graded, degrees of generators: [ -1, 0, 1 ])
```

```
gap> Display( n );
x^2,0,0,
x*y,0,0,
y^2,0,0,
0,  x,0,
0,  y,0,
0,  z,0,
0,  0,1

A left submodule generated by the 7 rows of the above matrix

(graded, degrees of generators: [ 1, 1, 1, 1, 1, 1, 1 ])
gap> N := UnderlyingObject( n );
<A graded left module presented by yet unknown relations for 7 generators>
gap> Display( N );
z, 0, 0,0,    0,  0,0,
0, z, 0,0,    0,  0,0,
0, 0, z,0,    0,  0,0,
0, 0, 0,0,    -z, y,0,
x, 0, 0,0,    y,  0,z,
-y,x, 0,0,    0,  0,0,
0, -y,x,0,    0,  0,0,
0, 0, 0,-y,   x,  0,0,
0, 0, 0,-z,   0,  x,0,
0, 0, 0,0,    y*z,0,z^2,
0, 0, 0,y^2*z,0,  0,x*z^2

Cokernel of the map

Q[x,y,z]^(1x11) --> Q[x,y,z]^(1x7),

currently represented by the above matrix

(graded, degrees of generators: [ 1, 1, 1, 1, 1, 1, 1 ])
gap> gens := GeneratorsOfModule( N );
<A set of 7 generators of a homalg left module>
gap> Display( gens );
x^2,0,0,
x*y,0,0,
y^2,0,0,
0,  x,0,
0,  y,0,
0,  z,0,
0,  0,1

a set of 7 generators given by the rows of the above matrix
```

5.6.5 RepresentationMapOfRingElement

▷ RepresentationMapOfRingElement(r, M, d)   (operation)
   Returns: a homalg matrix
   The graded map induced by the homogeneous degree 1 ring element r (of the underlying homalg

graded ring S) regarded as a R-linear map between the d-th and the $(d+1)$-st homogeneous part of the graded finitely generated homalg S-module M, where R is the coefficients ring of the graded ring S with $S_0 = R$. The generating set of both modules is given by GeneratorsOfHomogeneousPart (5.6.3). The entries of the matrix presenting the map lie in the coefficients ring R.

```
─────────────────────────── Example ───────────────────────────
  gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
  gap> S := GradedRing( R );;
  gap> x := Indeterminate( S, 1 );
  x
  gap> M := HomalgMatrix( "[ x^3, y^2, z,   z, 0, 0 ]", 2, 3, S );;
  gap> M := LeftPresentationWithDegrees( M, [ -1, 0, 1 ] );
  <A graded non-torsion left module presented by 2 relations for 3 generators>
  gap> m := RepresentationMapOfRingElement( x, M, 0 );
  <A "homomorphism" of graded left modules>
  gap> Display( m );
  1,0,0,0,0,0,0,
  0,1,0,0,0,0,0,
  0,0,0,1,0,0,0

  the graded map is currently represented by the above 3 x 7 matrix

  (degrees of generators of target: [ 1, 1, 1, 1, 1, 1, 1 ])
```

5.6.6   RepresentationMatrixOfKoszulId

▷ RepresentationMatrixOfKoszulId(d, M)   (operation)
    Returns: a homalg matrix

It is assumed that all indeterminates of the underlying homalg graded ring S are of degree 1. The output is the homalg matrix of the multiplication map $\mathrm{Hom}(A, M_d) \to \mathrm{Hom}(A, M_{d+1})$, where A is the Koszul dual ring of S, defined using the operation KoszulDualRing.

```
─────────────────────────── Example ───────────────────────────
  gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
  gap> S := GradedRing( R );;
  gap> A := KoszulDualRing( S, "a,b,c" );;
  gap> M := HomalgMatrix( "[ x^3, y^2, z,   z, 0, 0 ]", 2, 3, S );;
  gap> M := LeftPresentationWithDegrees( M, [ -1, 0, 1 ] );
  <A graded non-torsion left module presented by 2 relations for 3 generators>
  gap> m := RepresentationMatrixOfKoszulId( 0, M );
  <An unevaluated 3 x 7 matrix over a graded ring>
  gap> Display( m );
  a,b,0,0,0,0,0,
  0,a,b,0,0,0,0,
  0,0,0,a,b,c,0
  (over a graded ring)
```

5.6.7   RepresentationMapOfKoszulId

▷ RepresentationMapOfKoszulId(d, M)   (operation)
    Returns: a homalg map

It is assumed that all indeterminates of the underlying homalg graded ring S are of degree 1. The output is the the multiplication map $\mathrm{Hom}(A, M_d) \to \mathrm{Hom}(A, M_{d+1})$, where A is the Koszul dual ring of S, defined using the operation `KoszulDualRing`.

```
—————————————————————————— Example ——————————————————————————————
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
gap> S := GradedRing( R );;
gap> A := KoszulDualRing( S, "a,b,c" );;
gap> M := HomalgMatrix( "[ x^3, y^2, z,   z, 0, 0 ]", 2, 3, S );;
gap> M := LeftPresentationWithDegrees( M, [ -1, 0, 1 ] );
<A graded non-torsion left module presented by 2 relations for 3 generators>
gap> m := RepresentationMapOfKoszulId( 0, M );
<A homomorphism of graded left modules>
gap> Display( m );
a,b,0,0,0,0,0,
0,a,b,0,0,0,0,
0,0,0,a,b,c,0

the graded map is currently represented by the above 3 x 7 matrix

(degrees of generators of target: [ 4, 4, 4, 4, 4, 4, 4 ])
```

5.6.8 KoszulRightAdjoint

▷ KoszulRightAdjoint(*M, degree_lowest, degree_highest*)   (operation)

    Returns: a homalg cocomplex

It is assumed that all indeterminates of the underlying homalg graded ring S are of degree 1. Compute the homalg A-cocomplex C of Koszul maps of the homalg S-module `M` ($\to$ `RepresentationMapOfKoszulId` (5.6.7)) in the [*degree_lowest* .. *degree_highest*]. The Castelnuovo-Mumford regularity of `M` is characterized as the highest degree d, such that C is not exact at d. A is the Koszul dual ring of S, defined using the operation `KoszulDualRing`.

```
—————————————————————————— Example ——————————————————————————————
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
gap> S := GradedRing( R );;
gap> A := KoszulDualRing( S, "a,b,c" );;
gap> M := HomalgMatrix( "[ x^3, y^2, z,   z, 0, 0 ]", 2, 3, S );;
gap> M := LeftPresentationWithDegrees( M, [ -1, 0, 1 ], S );
<A graded non-torsion left module presented by 2 relations for 3 generators>
gap> CastelnuovoMumfordRegularity( M );
1
gap> R := KoszulRightAdjoint( M, -5, 5 );
<A cocomplex containing 10 morphisms of graded left modules at degrees
[ -5 .. 5 ]>
gap> R := KoszulRightAdjoint( M, 1, 5 );
<An acyclic cocomplex containing
4 morphisms of graded left modules at degrees [ 1 .. 5 ]>
gap> R := KoszulRightAdjoint( M, 0, 5 );
<A cocomplex containing 5 morphisms of graded left modules at degrees
[ 0 .. 5 ]>
gap> R := KoszulRightAdjoint( M, -5, 5 );
<A cocomplex containing 10 morphisms of graded left modules at degrees
[ -5 .. 5 ]>
```

```
gap> H := Cohomology( R );
<A graded cohomology object consisting of 11 graded left modules at degrees
[ -5 .. 5 ]>
gap> ByASmallerPresentation( H );
<A non-zero graded cohomology object consisting of
11 graded left modules at degrees [ -5 .. 5 ]>
gap> Cohomology( R, -2 );
<A graded zero left module>
gap> Cohomology( R, -3 );
<A graded zero left module>
gap> Cohomology( R, -1 );
<A graded cyclic torsion-free non-free left module presented by 2 relations fo\
r a cyclic generator>
gap> Cohomology( R, 0 );
<A graded non-zero cyclic left module presented by 3 relations for a cyclic ge\
nerator>
gap> Cohomology( R, 1 );
<A graded non-zero cyclic left module presented by 2 relations for a cyclic ge\
nerator>
gap> Cohomology( R, 2 );
<A graded zero left module>
gap> Cohomology( R, 3 );
<A graded zero left module>
gap> Cohomology( R, 4 );
<A graded zero left module>
gap> Display( Cohomology( R, -1 ) );
Q{a,b,c}/< b, a >

(graded, degree of generator: 0)
gap> Display( Cohomology( R, 0 ) );
Q{a,b,c}/< c, b, a >

(graded, degree of generator: 0)
gap> Display( Cohomology( R, 1 ) );
Q{a,b,c}/< b, a >

(graded, degree of generator: 2)
```

5.6.9 HomogeneousPartOverCoefficientsRing

▷ HomogeneousPartOverCoefficientsRing(d, M)   (operation)

   Returns: a homalg module

   The degree d homogeneous part of the graded R-module M as a module over the coefficient ring or field of R.

```
——————————————— Example ———————————————
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
gap> S := GradedRing( R );;
gap> M := HomalgMatrix( "[ x, y^2, z^3 ]", 3, 1, S );;
gap> M := Subobject( M, ( 1 * S )^0 );
<A graded torsion-free (left) ideal given by 3 generators>
gap> CastelnuovoMumfordRegularity( M );
4
```

```
gap> M1 := HomogeneousPartOverCoefficientsRing( 1, M );
<A graded left vector space of dimension 1 on a free generator>
gap> gen1 := GeneratorsOfModule( M1 );
<A set consisting of a single generator of a homalg left module>
gap> Display( M1 );
Q^(1 x 1)

(graded, degree of generator: 1)
gap> M2 := HomogeneousPartOverCoefficientsRing( 2, M );
<A graded left vector space of dimension 4 on free generators>
gap> Display( M2 );
Q^(1 x 4)

(graded, degrees of generators: [ 2, 2, 2, 2 ])
gap> gen2 := GeneratorsOfModule( M2 );
<A set of 4 generators of a homalg left module>
gap> M3 := HomogeneousPartOverCoefficientsRing( 3, M );
<A graded left vector space of dimension 9 on free generators>
gap> Display( M3 );
Q^(1 x 9)

(graded, degrees of generators: [ 3, 3, 3, 3, 3, 3, 3, 3, 3 ])
gap> gen3 := GeneratorsOfModule( M3 );
<A set of 9 generators of a homalg left module>
gap> Display( gen1 );
x

a set consisting of a single generator given by (the row of) the above matrix
gap> Display( gen2 );
x^2,
x*y,
x*z,
y^2

a set of 4 generators given by the rows of the above matrix
gap> Display( gen3 );
x^3,
x^2*y,
x^2*z,
x*y*z,
x*z^2,
x*y^2,
y^3,
y^2*z,
z^3

a set of 9 generators given by the rows of the above matrix
```

Chapter 6

The Tate Resolution

6.1 The Tate Resolution: Operations and Functions

6.1.1 TateResolution

▷ TateResolution(*M, degree_lowest, degree_highest*)   (operation)

   Returns: a homalg cocomplex

   Compute the Tate resolution of the sheaf *M*.

```
——————————————— Example ———————————————
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x0..x3";;
gap> S := GradedRing( R );;
gap> A := KoszulDualRing( S, "e0..e3" );;
```

In the following we construct the different exterior powers of the cotangent bundle shifted by 1.
Observe how a single 1 travels along the diagnoal in the window $[-3..0]x[0..3]$.

First we start with the structure sheaf with its Tate resolution:

```
——————————————— Example ———————————————
gap> O := S^0;
<The graded free left module of rank 1 on a free generator>
gap> T := TateResolution( O, -5, 5 );
<An acyclic cocomplex containing
10 morphisms of graded left modules at degrees [ -5 .. 5 ]>
gap> betti := BettiTable( T );
<A Betti diagram of <An acyclic cocomplex containing
10 morphisms of graded left modules at degrees [ -5 .. 5 ]>>
gap> Display( betti );
total:   35  20  10   4   1   1   4  10  20  35  56   ?   ?   ?
----------|---|---|---|---|---|---|---|---|---|---|---|---|---|
    3:   35  20  10   4   1   .   .   .   .   .   .   0   0   0
    2:    *   .   .   .   .   .   .   .   .   .   .   .   0   0
    1:    *   *   .   .   .   .   .   .   .   .   .   .   .   0
    0:    *   *   *   .   .   .   .   .   1   4  10  20  35  56
----------|---|---|---|---|---|---|---|---S---|---|---|---|---|
twist:   -8  -7  -6  -5  -4  -3  -2  -1   0   1   2   3   4   5
   -------------------------------------------------------------
Euler:  -35 -20 -10  -4  -1   0   0   0   1   4  10  20  35  56
```

The Castelnuovo-Mumford regularity of the *underlying module* is distinguished among the list of twists by the character 'V' pointing to it. It is *not* an invariant of the sheaf (see the next diagram).

The residue class field (i.e. S modulo the maximal homogeneous ideal):

```
─────────────────── Example ───────────────────
 gap> k := HomalgMatrix( Indeterminates( S ), Length( Indeterminates( S ) ), 1, S );
 <A 4 x 1 matrix over a graded ring>
 gap> k := LeftPresentationWithDegrees( k );
 <A graded cyclic left module presented by 4 relations for a cyclic generator>
```

Another way of constructing the structure sheaf:

```
─────────────────── Example ───────────────────
 gap> U0 := SyzygiesObject( 1, k );
 <A graded torsion-free left module presented by yet unknown relations for 4 ge\
 nerators>
 gap> T0 := TateResolution( U0, -5, 5 );
 <An acyclic cocomplex containing
 10 morphisms of graded left modules at degrees [ -5 .. 5 ]>
 gap> betti0 := BettiTable( T0 );
 <A Betti diagram of <An acyclic cocomplex containing
 10 morphisms of graded left modules at degrees [ -5 .. 5 ]>>
 gap> Display( betti0 );
 total:  35  20  10   4   1   1   4  10  20  35  56   ?   ?   ?
 ----------|---|---|---|---|---|---|---|---|---|---|---|---|---|
     3:    35  20  10   4   1   .   .   .   .   .   .   0   0   0
     2:     *   .   .   .   .   .   .   .   .   .   .   .   0   0
     1:     *   *   .   .   .   .   .   .   .   .   .   .   .   0
     0:     *   *   *   .   .   .   .   .   .   1   4  10  20  35  56
 ----------|---|---|---|---|---|---|---|---|---S---|---|---|---|---|
 twist:    -8  -7  -6  -5  -4  -3  -2  -1   0   1   2   3   4   5
 -------------------------------------------------------------
 Euler:  -35 -20 -10  -4  -1   0   0   0   1   4  10  20  35  56
```

The cotangent bundle:

```
─────────────────── Example ───────────────────
 gap> cotangent := SyzygiesObject( 2, k );
 <A graded torsion-free left module presented by yet unknown relations for 6 ge\
 nerators>
 gap> IsFree( UnderlyingModule( cotangent ) );
 false
 gap> Rank( cotangent );
 3
 gap> cotangent;
 <A graded reflexive non-projective rank 3 left module presented by 4 relations\
  for 6 generators>
 gap> ProjectiveDimension( UnderlyingModule( cotangent ) );
 2
```

the cotangent bundle shifted by 1 with its Tate resolution:

```
─────────────────── Example ───────────────────
 gap> U1 := cotangent * S^1;
 <A graded non-torsion left module presented by 4 relations for 6 generators>
```

```
gap> T1 := TateResolution( U1, -5, 5 );
<An acyclic cocomplex containing
10 morphisms of graded left modules at degrees [ -5 .. 5 ]>
gap> betti1 := BettiTable( T1 );
<A Betti diagram of <An acyclic cocomplex containing
10 morphisms of graded left modules at degrees [ -5 .. 5 ]>>
gap> Display( betti1 );
total:   120   70   36   15    4    1    6   20   45   84  140    ?    ?    ?
-----------|----|----|----|----|----|----|----|----|----|----|----|----|----|
     3:   120   70   36   15    4    .    .    .    .    .    .    0    0    0
     2:     *    .    .    .    .    .    .    .    .    .    .    .    0    0
     1:     *    *    .    .    .    .    .    1    .    .    .    .    .    0
     0:     *    *    *    .    .    .    .    .    .    6   20   45   84  140
-----------|----|----|----|----|----|----|----|----|----S----|----|----|----|
twist:    -8   -7   -6   -5   -4   -3   -2   -1    0    1    2    3    4    5
-------------------------------------------------------------------------------
Euler:  -120  -70  -36  -15   -4    0    0   -1    0    6   20   45   84  140
```

The second power U^2 of the shifted cotangent bundle $U = U^1$ and its Tate resolution:

```
───────── Example ─────────
gap> U2 := SyzygiesObject( 3, k ) * S^2;
<A graded rank 3 left module presented by 1 relation for 4 generators>
gap> T2 := TateResolution( U2, -5, 5 );
<An acyclic cocomplex containing
10 morphisms of graded left modules at degrees [ -5 .. 5 ]>
gap> betti2 := BettiTable( T2 );
<A Betti diagram of <An acyclic cocomplex containing
10 morphisms of graded left modules at degrees [ -5 .. 5 ]>>
gap> Display( betti2 );
total:   140   84   45   20    6    1    4   15   36   70  120    ?    ?    ?
-----------|----|----|----|----|----|----|----|----|----|----|----|----|----|
     3:   140   84   45   20    6    .    .    .    .    .    .    0    0    0
     2:     *    .    .    .    .    .    1    .    .    .    .    .    0    0
     1:     *    *    .    .    .    .    .    .    .    .    .    .    .    0
     0:     *    *    *    .    .    .    .    .    .    4   15   36   70  120
-----------|----|----|----|----|----|----|----|----|----S----|----|----|----|
twist:    -8   -7   -6   -5   -4   -3   -2   -1    0    1    2    3    4    5
-------------------------------------------------------------------------------
Euler:  -140  -84  -45  -20   -6    0    1    0    0    4   15   36   70  120
```

The third power U^3 of the shifted cotangent bundle $U = U^1$ and its Tate resolution:

```
───────── Example ─────────
gap> U3 := SyzygiesObject( 4, k ) * S^3;
<A graded free left module of rank 1 on a free generator>
gap> Display( U3 );
Q[x0,x1,x2,x3]^(1 x 1)

(graded, degree of generator: 1)
gap> T3 := TateResolution( U3, -5, 5 );
<An acyclic cocomplex containing
10 morphisms of graded left modules at degrees [ -5 .. 5 ]>
gap> betti3 := BettiTable( T3 );
```

```
  <A Betti diagram of <An acyclic cocomplex containing
  10 morphisms of graded left modules at degrees [ -5 .. 5 ]>>
  gap> Display( betti3 );
  total:  56  35  20  10   4   1   1   4  10  20  35   ?   ?   ?
  ----------|---|---|---|---|---|---|---|---|---|---|---|---|---|
      3:    56  35  20  10   4   1   .   .   .   .   .   0   0   0
      2:     *   .   .   .   .   .   .   .   .   .   .   .   0   0
      1:     *   *   .   .   .   .   .   .   .   .   .   .   .   0
      0:     *   *   *   .   .   .   .   .   .   1   4  10  20  35
  ----------|---|---|---|---|---|---|---|---|---|---S---|---|---|
  twist:    -8  -7  -6  -5  -4  -3  -2  -1   0   1   2   3   4   5
  ---------------------------------------------------------------
  Euler:   -56 -35 -20 -10  -4  -1   0   0   0   1   4  10  20  35
```

Another way to construct $U^2 = U^{(3-1)}$:

```
                            —— Example ——
  gap> u2 := GradedHom( U1, S^(-1) );
  <A graded torsion-free right module on 4 generators satisfying yet unknown rel\
  ations>
  gap> t2 := TateResolution( u2, -5, 5 );
  <An acyclic cocomplex containing
  10 morphisms of graded right modules at degrees [ -5 .. 5 ]>
  gap> BettiTable( t2 );
  <A Betti diagram of <An acyclic cocomplex containing
  10 morphisms of graded right modules at degrees [ -5 .. 5 ]>>
  gap> Display( last );
  total:  140  84  45  20   6   1   4  15  36   70  120   ?   ?   ?
  -----------|----|----|----|----|----|----|----|----|----|----|----|----|----|
      3:     140  84  45  20   6   .   .   .   .    .    .   0   0   0
      2:      *   .   .   .   .   .   1   .   .    .    .   0   0
      1:      *   *   .   .   .   .   .   .   .    .    .   .   0
      0:      *   *   *   .   .   .   .   .   .    4   15  36  70 120
  -----------|----|----|----|----|----|----|----|----|----S----|----|----|----|
  twist:      -8  -7  -6  -5  -4  -3  -2  -1   0    1    2   3   4   5
  ---------------------------------------------------------------------------
  Euler:    -140 -84 -45 -20  -6   0   1   0   0    4   15  36  70 120
```

Chapter 7

Examples

7.1 Betti Diagrams

7.1.1 DE-2.2

```
                         ────── Example ──────
 gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x0,x1,x2";;
 gap> S := GradedRing( R );;
 gap> mat := HomalgMatrix( "[ x0^2, x1^2, x2^2 ]", 1, 3, S );
 <A 1 x 3 matrix over a graded ring>
 gap> M := RightPresentationWithDegrees( mat, S );
 <A graded cyclic right module on a cyclic generator satisfying 3 relations>
 gap> M := RightPresentationWithDegrees( mat );
 <A graded cyclic right module on a cyclic generator satisfying 3 relations>
 gap> d := Resolution( M );
 <A right acyclic complex containing
 3 morphisms of graded right modules at degrees [ 0 .. 3 ]>
 gap> betti := BettiTable( d );
 <A Betti diagram of <A right acyclic complex containing
 3 morphisms of graded right modules at degrees [ 0 .. 3 ]>>
 gap> Display( betti );
  total:  1 3 3 1
 ----------------
      0:  1 . . .
      1:  . 3 . .
      2:  . . 3 .
      3:  . . . 1
 ----------------
 degree:  0 1 2 3
 gap> ## we are still below the Castelnuovo-Mumford regularity, which is 3:
 gap> M2 := SubmoduleGeneratedByHomogeneousPart( 2, M );
 <A graded torsion right submodule given by 3 generators>
 gap> d2 := Resolution( M2 );
 <A right acyclic complex containing
 3 morphisms of graded right modules at degrees [ 0 .. 3 ]>
 gap> betti2 := BettiTable( d2 );
 <A Betti diagram of <A right acyclic complex containing
 3 morphisms of graded right modules at degrees [ 0 .. 3 ]>>
 gap> Display( betti2 );
  total:  3 8 6 1
```

```
    ---------------
     2:  3 8 6 .
     3:  . . . 1
    ---------------
    degree:  0 1 2 3
```

7.1.2 DE-Code

```
────────────────────────── Example ──────────────────────────
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x0,x1,x2";;
gap> S := GradedRing( R );;
gap> mat := HomalgMatrix( "[ x0^2, x1^2 ]", 1, 2, S );
<A 1 x 2 matrix over a graded ring>
gap> M := RightPresentationWithDegrees( mat, S );
<A graded cyclic right module on a cyclic generator satisfying 2 relations>
gap> d := Resolution( M );
<A right acyclic complex containing
2 morphisms of graded right modules at degrees [ 0 .. 2 ]>
gap> betti := BettiTable( d );
<A Betti diagram of <A right acyclic complex containing
2 morphisms of graded right modules at degrees [ 0 .. 2 ]>>
gap> Display( betti );
 total:  1 2 1
--------------
     0:  1 . .
     1:  . 2 .
     2:  . . 1
--------------
degree:  0 1 2
gap> m := SubmoduleGeneratedByHomogeneousPart( 2, M );
<A graded torsion right submodule given by 4 generators>
gap> d2 := Resolution( m );
<A right acyclic complex containing
2 morphisms of graded right modules at degrees [ 0 .. 2 ]>
gap> betti2 := BettiTable( d2 );
<A Betti diagram of <A right acyclic complex containing
2 morphisms of graded right modules at degrees [ 0 .. 2 ]>>
gap> Display( betti2 );
     2:  4 8 4
--------------
degree:  0 1 2
```

7.1.3 Schenck-3.2

This is an example from Section 3.2 in [Sch03].

```
────────────────────────── Example ──────────────────────────
gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
gap> mmat := HomalgMatrix( "[ x, x^3 + y^3 + z^3 ]", 1, 2, Qxyz );
<A 1 x 2 matrix over an external ring>
gap> S := GradedRing( Qxyz );;
gap> M := RightPresentationWithDegrees( mmat, S );
<A graded cyclic right module on a cyclic generator satisfying 2 relations>
```

```
gap> Mr := Resolution( M );
<A right acyclic complex containing
2 morphisms of graded right modules at degrees [ 0 .. 2 ]>
gap> bettiM := BettiTable( Mr );
<A Betti diagram of <A right acyclic complex containing
2 morphisms of graded right modules at degrees [ 0 .. 2 ]>>
gap> Display( bettiM );
 total:  1 2 1
 -------------
     0:  1 1 .
     1:  . . .
     2:  . 1 1
 -------------
degree:  0 1 2
gap> R := GradedRing( CoefficientsRing( S ) * "x,y,z,w" );;
gap> nmat := HomalgMatrix( "[ z^2 - y*w, y*z - x*w, y^2 - x*z ]", 1, 3, R );
<A 1 x 3 matrix over a graded ring>
gap> N := RightPresentationWithDegrees( nmat );
<A graded cyclic right module on a cyclic generator satisfying 3 relations>
gap> Nr := Resolution( N );
<A right acyclic complex containing
2 morphisms of graded right modules at degrees [ 0 .. 2 ]>
gap> bettiN := BettiTable( Nr );
<A Betti diagram of <A right acyclic complex containing
2 morphisms of graded right modules at degrees [ 0 .. 2 ]>>
gap> Display( bettiN );
 total:  1 3 2
 -------------
     0:  1 . .
     1:  . 3 2
 -------------
degree:  0 1 2
```

7.1.4 Schenck-8.3

This is an example from Section 8.3 in [Sch03].

```
                          —— Example ——
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z,w";;
gap> S := GradedRing( R );;
gap> jmat := HomalgMatrix( "[ z*w, x*w, y*z, x*y, x^3*z - x*z^3 ]", 1, 5, S );
<A 1 x 5 matrix over a graded ring>
gap> J := RightPresentationWithDegrees( jmat );
<A graded cyclic right module on a cyclic generator satisfying 5 relations>
gap> Jr := Resolution( J );
<A right acyclic complex containing
3 morphisms of graded right modules at degrees [ 0 .. 3 ]>
gap> betti := BettiTable( Jr );
<A Betti diagram of <A right acyclic complex containing
3 morphisms of graded right modules at degrees [ 0 .. 3 ]>>
gap> Display( betti );
 total:  1 5 6 2
 ---------------
```

```
      0:  1 . . .
      1:  . 4 4 1
      2:  . . . .
      3:  . 1 2 1
----------------
degree:  0 1 2 3
```

7.1.5   Schenck-8.3.3

This is Exercise 8.3.3 in [Sch03].

```
────────────── Example ──────────────
 gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
 gap> S := GradedRing( Qxyz );;
 gap> mat := HomalgMatrix( "[ x*y*z, x*y^2, x^2*z, x^2*y, x^3 ]", 1, 5, S );
 <A 1 x 5 matrix over a graded ring>
 gap> M := RightPresentationWithDegrees( mat, S );
 <A graded cyclic right module on a cyclic generator satisfying 5 relations>
 gap> Mr := Resolution( M );
 <A right acyclic complex containing
 3 morphisms of graded right modules at degrees [ 0 .. 3 ]>
 gap> betti := BettiTable( Mr );
 <A Betti diagram of <A right acyclic complex containing
 3 morphisms of graded right modules at degrees [ 0 .. 3 ]>>
 gap> Display( betti );
  total:  1 5 6 2
 ----------------
      0:  1 . . .
      1:  . . . .
      2:  . 5 6 2
 ----------------
 degree:  0 1 2 3
```

7.2   Commutative Algebra

7.2.1   Saturate

```
────────────── Example ──────────────
 gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";;
 gap> S := GradedRing( R );;
 gap> m := GradedLeftSubmodule( "x,y,z", S );
 <A graded torsion-free (left) ideal given by 3 generators>
 gap> I := Intersect( m^3, GradedLeftSubmodule( "x", S ) );
 <A graded torsion-free (left) ideal given by 6 generators>
 gap> NrRelations( I );
 8
 gap> Im := SubobjectQuotient( I, m );
 <A graded torsion-free rank 1 (left) ideal given by 3 generators>
 gap> I_m := Saturate( I, m );
 <A graded principal (left) ideal of rank 1 on a free generator>
 gap> Is := Saturate( I );
 <A graded principal (left) ideal of rank 1 on a free generator>
 gap> Assert( 0, Is = I_m );
```

7.3 Global Section Modules of the Induced Sheaves

7.3.1 Examples of the ModuleOfGlobalSections Functor and Purity Filtrations

```
———————————————— Example ————————————————
 gap> LoadPackage( "GradedRingForHomalg" );;
 gap> Qxyzt := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z,t";;
 gap> S := GradedRing( Qxyzt );;
 gap>
 gap> wmat := HomalgMatrix( "[ \
 > x*y,  y*z,    z*t,       0,           0,         0,\
 > x^3*z,x^2*z^2,0,         x*z^2*t,     -z^2*t^2,  0,\
 > x^4,  x^3*z,  0,         x^2*z*t,     -x*z*t^2,  0,\
 > 0,    0,      x*y,       -y^2,        x^2-t^2,   0,\
 > 0,    0,      x^2*z,     -x*y*z,      y*z*t,     0,\
 > 0,    0,      x^2*y-x^2*t,-x*y^2+x*y*t,y^2*t-y*t^2,0,\
 > 0,    0,      0,         0,           -1,        1 \
 > ]", 7, 6, Qxyzt );;
 gap>
 gap> LoadPackage( "GradedModules" );;
 gap> wmor := GradedMap( wmat, "free", "free", "left", S );;
 gap> IsMorphism( wmor );;
 gap> W := LeftPresentationWithDegrees( wmat, S );;
 gap> HW := ModuleOfGlobalSections( W );
 <A graded left module presented by yet unknown relations for 6 generators>
 gap> LinearStrandOfTateResolution( W, 0,4 );
 <A cocomplex containing 4 morphisms of graded left modules at degrees
 [ 0 .. 4 ]>
 gap> purity_iso := IsomorphismOfFiltration( PurityFiltration( W ) );
 <A non-zero isomorphism of graded left modules>
 gap> Hpurity_iso := ModuleOfGlobalSections( purity_iso );
 <An isomorphism of graded left modules>
 gap> ModuleOfGlobalSections( wmor );
 <A homomorphism of graded left modules>
 gap> NaturalMapToModuleOfGlobalSections( W );
 <A homomorphism of graded left modules>
```

7.3.2 Horrocks Mumford bundle

This example computes the global sections module of the Horrocks-Mumford bundle.

```
———————————————— Example ————————————————
 gap> LoadPackage( "GradedRingForHomalg" );;
 gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x0..x4";;
 gap> S := GradedRing( R );;
 gap> A := KoszulDualRing( S, "e0..e4" );;
 gap> LoadPackage( "GradedModules" );;
 gap> mat := HomalgMatrix( "[ \
 > e1*e4, e2*e0, e3*e1, e4*e2, e0*e3, \
 > e2*e3, e3*e4, e4*e0, e0*e1, e1*e2  \
 > ]",
 > 2, 5, A );
 <A 2 x 5 matrix over a graded ring>
 gap> phi := GradedMap( mat, "free", "free", "left", A );;
```

```
gap> IsMorphism( phi );
true
gap> M := GuessModuleOfGlobalSectionsFromATateMap( 2, phi );
#I  GuessModuleOfGlobalSectionsFromATateMap uses a heuristic for efficiency;
please check the correctness of the following result

<A graded left module presented by yet unknown relations for 19 generators>
gap> IsPure( M );
true
gap> Rank( M );
2
gap> Display( BettiTable( Resolution( M ) ) );
 total:  19 35 20  2
-------------------
     3:   4  .  .  .
     4:  15 35 20  .
     5:   .  .  .  2
-------------------
degree:  0  1  2  3
gap> Display( BettiTable( TateResolution( M, -5, 5 ) ) );
total: 100 37 14 10  5  2  5 10 14 37 100  ?  ?  ?  ?
----------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
     4: 100 35  4  .  .  .  .  .  .  .  .  0  0  0  0
     3:   *  .  2 10 10  5  .  .  .  .  .  .  0  0  0
     2:   *  *  .  .  .  .  .  2  .  .  .  .  .  0  0
     1:   *  *  *  .  .  .  .  .  .  5 10 10  2  .  0
     0:   *  *  *  *  .  .  .  .  .  .  .  .  .  4 35 100
----------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---S
twist:  -9 -8 -7 -6 -5 -4 -3 -2 -1  0  1  2  3  4  5
--------------------------------------------------------------------
Euler: 100 35  2 -10 -10 -5  0  2  0 -5 -10 -10  2 35 100
gap> M;
<A graded reflexive non-projective rank 2 left module presented by 94 \
relations for 19 generators>
gap> P := ElementOfGrothendieckGroup( M );
( 2*O_{P^4} - 1*O_{P^3} - 4*O_{P^2} - 2*O_{P^1} ) -> P^4
gap> P!.DisplayTwistedCoefficients := true;
true
gap> P;
( 2*O(-3) - 10*O(-2) + 15*O(-1) - 5*O(0) ) -> P^4
gap> chi := HilbertPolynomial( M );
1/12*t^4+2/3*t^3-1/12*t^2-17/3*t-5
gap> c := ChernPolynomial( M );
( 2 | 1-h+4*h^2 ) -> P^4
gap> ChernPolynomial( M * S^3 );
( 2 | 1+5*h+10*h^2 ) -> P^4
gap> ch := ChernCharacter( M );
[ 2-u-7*u^2/2!+11*u^3/3!+17*u^4/4! ] -> P^4
gap> HilbertPolynomial( ch );
1/12*t^4+2/3*t^3-1/12*t^2-17/3*t-5
gap> List( [ -8 .. 7 ], i -> Value( chi, i ) );
[ 35, 2, -10, -10, -5, 0, 2, 0, -5, -10, -10, 2, 35, 100, 210, 380 ]
gap> HF := HilbertFunction( M );
```

```
function( t ) ... end
gap> List( [ 0 .. 7 ], HF );
[ 0, 0, 0, 4, 35, 100, 210, 380 ]
gap> IndexOfRegularity( M );
4
gap> DataOfHilbertFunction( M );
[ [ [ 4 ], [ 3 ] ], 1/12*t^4+2/3*t^3-1/12*t^2-17/3*t-5 ]
```

Appendix A

Overview of the **GradedModules** Package Source Code

References

[hpa10] The homalg project authors. *The homalg project*, 2003-2010. http://homalg.math.rwth-aachen.de/. 4

[Sch03] Hal Schenck. *Computational algebraic geometry*, volume 58 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2003. 21, 22, 23

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